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Gauge symmetry in Lagrangian formulation and Schwinger models

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Abstract. A systematic method is presented for processing the equations of motion of a singular Lagrangian that, in principle, has all dynamical degrees of freedom together with the Lagrangian constraints and a number of identities between the equations of motion. Then, using these identities one can find the complete set of gauge transformations of the system. Different types of Schwinger model are considered as examples and the related gauge transformations are derived.

1. Introduction

Gauge invariance is one of the most fundamental concepts in modern theoretical physics. In spite of its fame, however, the exact meaning of gauge invariance is not well defined. For particle physicists, it means local phase transformation of matter fields followed by appropriate transformation of gauge fields. However, gauge symmetry has some other interpretations from a more general point of view. In some references it is defined as any transformation involving arbitrary functions of time which maps solutions of the equations of motion into each other [1, 2].

Another definition [3] which is as good, is that a gauge transformation (GT) is any transformation, involving an arbitrary function of time, which does not change the action. By this, we mean that if δq_i is the GT of an arbitrary trajectory $q(t)$ then

$$S[q(t)] = S[q(t) + \delta q(t)]. \tag{1}$$

In the special case that $q_0(t)$ is the trajectory of the classical system, i.e. makes $S[q_0(t)]$ stationary, then $q_0(t) + \delta q(t)$ is also another stationary point of the action, provided we adjust $\delta q(t)$ such that it vanishes at the endpoints. So $q_0'(t) = q_0(t) + \delta q(t)$ is also another classical trajectory. By this definition a GT also has the property of mapping solution of equations of motion into each other.

Given an arbitrary action, the following questions arise.

(1) Is there any gauge symmetry in the model?

(2) What are the exact forms of the GTs?

(3) How many dynamical, constraint and gauge degrees of freedom are there in the system?

The gauge symmetry is often put, by hand, in the action while constructing the Lagrangian, and sometimes it is found by direct observation or trial and error. However, it

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may be, that for some complicated Lagrangians the gauge symmetry cannot be seen directly. An example of this type is referred to in [3]. So, in general, one needs a systematic method to answer the above questions.

Since every gauge theory is related to a Hamiltonian constrained system [4], several authors have tried to find the answer within the framework of a Hamilton–Dirac formulation [5]. For example the famous Dirac conjecture, about the generators of GTs, has been much discussed [6–8]. The necessary and sufficient conditions for a function in phase space to be the generator of a GT is discussed in [1, 6]. In [2] it is proved that there exist some appropriate chains of constraints that can be used to construct the most general form of the generator of GTs. [3] gives an algorithm for obtaining a complete set of gauge symmetries, using the framework of Hamiltonian formulation.

On the other hand little has been achieved in the Lagrangian formalism. As is well known, for every gauge-invariant theory the Euler–Lagrange equations of motion cannot be solved completely to determine all the accelerations [9]. In other words, undetermined accelerations imply the appearance of arbitrary functions of time, which is the signature of gauge theories.

It is the aim of this paper to propose a systematic method for deriving the explicit form of the GTs from the Lagrangian equations of motion. In section 2, we propose a procedure to find the greatest number of equations including accelerations for a singular Lagrangian by means of differentiating the Lagrangian constraints. On the other hand a number of identities among the Euler derivatives emerge. We show how in principle it is possible to recognize and count different types of degrees of freedom; these are gauge, dynamical and constraint ones. We do not make any special assumption about the Lagrangian, and consider the most general feature of the problem.

In section 3, we show that for each of the above mentioned identities one can find a transformation of the coordinates involving an arbitrary function of time which does not change the action, i.e. a GT. Some remarks about the gauge-invariant field theories and application of these methods are discussed in section 3. Section 4 is related to the Schwinger model. The model is discussed in the framework of Hamiltonian constraint systems in several texts [10–13]. Here we obtain the gauge symmetries of the ordinary and axial Schwinger models by using our Lagrangian method and also discuss the equations of motion of the generalized and chiral Schwinger models.

2. Dynamics of a singular Lagrangian

Consider a dynamical system with k degrees of freedom, described by the Lagrangian $L(q,\dot{q})$. The equations of motion can be obtained by vanishing the Euler derivatives as follows,

$$L_i \equiv W_{ij}\ddot{q}_j + \alpha_i = 0 \qquad i = 1, \dots, k \tag{2}$$

where the elements of Hessian matrix W, and the α_i are defined as

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \tag{3}$$

$$\alpha_i = \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\dot{q}_j - \frac{\partial L}{\partial q_i}. \tag{4}$$

We assume summation over the repeated indices throughout this paper. For a singular Lagrangian the determinant of W vanishes, and therefore the equations of motion (2) cannot be solved for all of the accelerations $\ddot{q}_j$. Suppose the rank of W is $(k - A_1)$, so there are A_1 null eigenvectors λ^{a_1} for W such that

$$\lambda_i^{a_1} W_{ij} = 0 \qquad a_1 = 1, \ldots, A_1. \tag{5}$$

If we multiply both sides of equations (2) by λ^{a_1} we obtain

$$\gamma^{a_1}(q, \dot{q}) = \lambda_i^{a_1} L_i = \lambda_i^{a_1} \alpha_i = 0 \qquad a_1 = 1, \ldots, A_1. \tag{6}$$

The functions γ^{a_1} are A_1 constraints of velocities and coordinates; but they are not necessarily independent of each other. suppose the rank of equations (6) is $\bar{A}_1$. It means that one can, in principle, find $\bar{A}_1$ independent functions $\gamma^{\bar{a}_1}(q, \dot{q})$ such that their vanishing is the necessary and sufficient condition for the vanishing of γ^{a_1}. In other words, γ^{a_1} are weakly vanishing functions on the surface $\gamma^{\bar{a}_1}(q, \dot{q}) = 0$.

So the $\gamma^{\bar{a}_1}$ are independent linear combinations of γ^{a_1} with coefficients which may depend on q_i and $\dot{q}_i$:

$$\gamma^{\bar{a}_1}(q, \dot{q}) = \sum_{a_1=1}^{A_1} C_{a_1}^{\bar{a}_1}(q, \dot{q}) \gamma^{a_1}(q, \dot{q}) \qquad \bar{a}_1 = 1, \ldots, \bar{A}_1. \tag{7}$$

On the other hand there are $\hat{A}_1 = A_1 - \bar{A}_1$ linear combinations of γ^{a_1} which vanish identically:

$$\sum_{a_1=1}^{A_1} C_{a_1}^{\hat{a}_1}(q, \dot{q}) \gamma^{a_1}(q, \dot{q}) = 0 \qquad \hat{a}_1 = 1, \ldots, \hat{A}_1. \tag{8}$$

Comparing (7) with (6) shows that using the null eigenvectors

$$\lambda^{\bar{a}_1}(q, \dot{q}) = \sum_{a_1=1}^{A_1} C_{a_1}^{\bar{a}_1}(q, \dot{q}) \lambda^{a_1}(q, \dot{q}) \qquad \bar{a}_1 = 1, \ldots, \bar{A}_1 \tag{9}$$

the primary Lagrangian constraints can be written as,

$$\gamma^{\bar{a}_1}(q, \dot{q}) = \lambda_i^{\bar{a}_1} L_i \qquad \bar{a}_1 = 1, \ldots, \bar{A}_1. \tag{10}$$

Similarly, comparing (8) with (6) shows that for the null eigenvectors

$$\lambda^{\hat{a}_1}(q, \dot{q}) = \sum_{a_1=1}^{A_1} C_{a_1}^{\hat{a}_1}(q, \dot{q}) \lambda^{a_1}(q, \dot{q}) \qquad \hat{a}_1 = 1, \ldots, \hat{A}_1 \tag{11}$$

the following identities can be written between Euler derivatives:

$$\lambda_i^{\hat{a}_1} L_i = 0 \qquad \hat{a}_1 = 1, \ldots, \hat{A}_1. \tag{12}$$

It should, however, be noted that a set of independent Lagrangian constraints are not necessarily independent functions of velocity. Here we consider only the so-called B-type constraints (in the terminology of [9]), for simplicity. This means that the constraints are independent functions of velocity. So we assume that the matrix $\partial \gamma^{\bar{a}_1} / \partial \dot{q}_j$ has the maximal rank $\bar{A}_1$. The treatment of the most general case, where the A-type constraints are also present, is given in the appendix.

In the next step using the consistency condition for the constraints with time, we add the time derivatives of the primary constraints (10) to the equations of motion (2). Therefore we have $k + \bar{A}_1$ equations which contain acceleration as follows

$$\begin{cases} W_{ij} \ddot{q}_j + \alpha_i = 0 & i = 1, \ldots, k \\ \dfrac{\mathrm{d}\gamma^{\bar{a}_1}}{\mathrm{d}t} = \dfrac{\partial \gamma^{\bar{a}_1}}{\partial \dot{q}_j} \ddot{q}_j + \dfrac{\partial \gamma^{\bar{a}_1}}{\partial q_j} \dot{q}_j = 0 & \bar{a}_1 = 1, \ldots, \bar{A}_1. \end{cases} \tag{13}$$

The added equations can be organized in such a manner that equations (13) read as

$$L^1_{i_1} \equiv W^1_{i_1 j}\ddot{q}_j + \alpha^1_{i_1} = 0 \qquad i_1 = 1, \ldots, k + \bar{A}_1 \tag{14}$$

where the first $kL^1_{i_1}$ are the same as L_i in equations (2) and the next $\bar{A}_1$ are $\frac{\mathrm{d}\gamma^{\bar{a}_1}}{\mathrm{d}t}$.

At this point we search for the left null eigenvectors of the rectangular matrix W^1. Since $\partial\gamma^{\bar{a}_1}/\partial\dot{q}_j$ has the maximal rank $\bar{A}_1$, no null eigenvector with vanishing first k components can be found. On the other hand if we add $\bar{A}_1$ zero components at the end of the previous null eigenvectors $\lambda^{\bar{a}_1}$ they are still null eigenvectors of $W^1_{i_1 j}$. However, there exist the possibility of finding some more (left) null eigenvectors for $W^1_{i_1 j}$ with some nonvanishing elements in the last $\bar{A}_1$ components and also in the first k components. We call them *new* null eigenvectors and denote them by λ^{a_2}. They should be considered modula the previous null eigenvectors λ^{a_1}, since each new null eigenvector combined with the previous ones also has the same properties.

Suppose there are A_2 ($A_2 \leqslant \bar{A}_1$) new null eigenvectors λ^{a_2} such that

$$\lambda^{a_2}_{i_1} W^1_{i_1 j} = 0 \qquad a_2 = 1, \ldots, A_2. \tag{15}$$

So, the rank of the set of equations (14) is,

$$(k - A_1) + \bar{A}_1 - A_2.$$

Again, multiplying both sides of equations (14) by λ^{a_2}, one obtains

$$\gamma^{a_2}(q, \dot{q}) = \lambda^{a_2}_{i_1} L^1_{i_1} = \lambda^{a_2}_{i_1}\alpha_{i_1} = 0 \qquad a_2 = 1, \ldots, A_2. \tag{16}$$

The *new* constraints $\gamma^{a_2}(q, \dot{q})$ are independent of the previous constraints $\gamma^{a_1}(q, \dot{q})$, as λ^{a_2} are calculated modula λ^{a_1}. However, as in the previous step, it is possible that they are not independent functions of coordinates and velocity. Again one can in principle find $\bar{A}_2$ independent functions $\gamma^{\bar{a}_2}(q, \dot{q})$ out of the γ^{a_2} as secondary Lagrangian constraints and $\hat{A}_2 = A_2 - \bar{A}_2$ identities between the $L^1_{i_1}$.

Suppose the null eigenvectors $\lambda^{\hat{a}_2}(q, \dot{q})$ are found in a similar way to relation (11) in the first step, such that the identities between the $L^1_{i_1}$ read as,

$$\lambda^{\hat{a}_2}_{i_1} L^1_{i_1} = 0 \qquad \hat{a}_2 = 1, \ldots, \hat{A}_2. \tag{17}$$

For future use, let us find the exact form of the identities (17). Using equations (13) and (11) we have,

$$\lambda^{\hat{a}_2}_i L_i + \lambda^{\hat{a}_2}_{\bar{a}_1} \frac{\mathrm{d}}{\mathrm{d}t}(\lambda^{\bar{a}_1}_i L_i) = 0 \qquad \hat{a}_2 = 1, \ldots, \hat{A}_2 \tag{18}$$

where the summations are over the appropriate domains. This result can be written in the form,

$$(\lambda^{\hat{a}_2}_i - \dot{\lambda}^{\hat{a}_2}_{\bar{a}_1}\lambda^{\bar{a}_1}_i)L_i + \frac{\mathrm{d}}{\mathrm{d}t}[(\lambda^{\hat{a}_2}_{\bar{a}_1}\lambda^{\bar{a}_1}_i)L_i] = 0 \qquad \hat{a}_2 = 1, \ldots, \hat{A}_2. \tag{19}$$

Similarly using (10), the secondary Lagrangian constraints are also in the form

$$\gamma^{\bar{a}_2}(q, \dot{q}) = (\lambda^{\bar{a}_2}_i - \dot{\lambda}^{\bar{a}_2}_{\bar{a}_1}\lambda^{\bar{a}_1}_i)L_i + \frac{\mathrm{d}}{\mathrm{d}t}[(\lambda^{\bar{a}_2}_{\bar{a}_1}\lambda^{\bar{a}_1}_i)L_i] = 0 \qquad \bar{a}_2 = 1, \ldots, \bar{A}_2. \tag{20}$$

Now the algorithm is clear. In the next step we should add the derivative of $\gamma^{\bar{a}_2}(q, \dot{q})$ to the set of equations (14), construct a longer matrix for coefficients of accelerations, namely $W^2_{i_2 j}(i_2 = 1, \ldots, k + \bar{A}_1 + \bar{A}_2)$ and search for some *new* null eigenvectors. We should proceed in this manner step by step. At each step some relations among the L_i and their derivatives as in (19) and some new constraints as in (20) emerge.

Consider, for example, the nth step, where we begin with the following $k+\bar{A}_1+\cdots+\bar{A}_n$ equations for acceleration:

$$\begin{cases} W_{ij}\ddot{q}_j + \alpha_i = 0 & i = 1, \ldots, k \\ \dfrac{\mathrm{d}\gamma^{\bar{a}_1}}{\mathrm{d}t} = 0 & \bar{a}_1 = 1, \ldots, \bar{A}_1 \\ \vdots & \\ \dfrac{\mathrm{d}\gamma^{\bar{a}_n}}{\mathrm{d}t} = 0 & \bar{a}_n = 1, \ldots, \bar{A}_n. \end{cases} \tag{21}$$

These can be summarized as,

$$W^n_{i_n j}\ddot{q}_j + \alpha_{i_n} = 0 \qquad i_n = 1, \ldots, k + \bar{A}_1 + \bar{A}_n. \tag{22}$$

In this step we have added $\bar{A}_n$ equations for accelerations. It may happen that there appear A_{n+1} *new* null eigenvectors $\lambda^{a_{n+1}}$ (modula the previous ones) for W^n, which have some nonvanishing elements in the first k and the last $\bar{A}_n$ components. So the total rank of equations (21) or (22) is,

$$(k - A_1) + (\bar{A}_1 - A_2) + \cdots + (\bar{A}_n - A_{n+1}). \tag{23}$$

Multiplying both sides of (22) by $\lambda^{a_{n+1}}_{i_n}$ leads to $\bar{A}_{n+1}$ $(n+1)$ary Lagrangian constraints $\gamma^{\bar{a}_{n+1}}$ plus $\hat{A}_{n+1}$ new relations among the L_i and their derivatives as follows:

$$\lambda^{\hat{a}_{n+1}}_i L_i + \lambda^{\hat{a}_{n+1}}_{\bar{a}_1}\frac{\mathrm{d}\gamma^{\bar{a}_1}}{\mathrm{d}t} + \cdots + \lambda^{\hat{a}_{n+1}}_{\bar{a}_n}\frac{\mathrm{d}\gamma^{\bar{a}_n}}{\mathrm{d}t} = 0. \tag{24}$$

Inserting $\gamma^{\bar{a}_n}$ in terms of L_i and their derivatives, as in the expressions in (20), and then constructing total derivatives, (24) can be written as,

$$\sum_{s=0}^{n} \frac{\mathrm{d}^s}{\mathrm{d}t^s}(\phi_{si}L_i) = 0. \tag{25}$$

In this relation ϕ_{si} are some functions of the coordinates and their derivatives emerging from $\lambda^{\bar{a}_m}$ with $m \leqslant n$ and $\lambda^{\hat{a}_{n+1}}$, like the expressions in (19).

It should be mentioned that for finding the vanishing combinations of $\gamma^{a_{n+1}}$, for example (8), one can use weakly vanishing ones. In this case, similar to the relations (11) and (12), we find some null eigenvectors $\lambda^{\hat{a}_{n+1}}$ for W^n such that

$$\lambda^{\hat{a}_{n+1}}_{i_n} L_{i_n} = \sum_{i\leqslant n} D_{\bar{a}_i}\gamma^{\bar{a}_i} \tag{26}$$

where $D_{\bar{a}_i}$ are some coefficients. The right-hand sides of (26) is a combination of previous L_{i_k} with $k \leqslant n$. So a suitable combination of the new null eigenvector $\lambda^{\hat{a}_{n+1}}$ with the previous ones would provide another null eigenvector $\lambda'^{\hat{a}_{n+1}}$ for which the relation $\lambda'^{\hat{a}_{n+1}}_{i_n} L_{i_n} = 0$ holds strongly. We assume that all these calculations have been performed at each stage.

Fortunately the story does end, since as we observed, at each step one loses a number of dynamical degrees of freedom. In other words, the total number of the null eigenvectors during all steps cannot exceed the number of degrees of freedom. Suppose the Nth step is the last one. There are two ways for this to happen. The first one is that no new null eigenvector can be found for W^N. This means that $A_{N+1} = 0$. Therefore from (23) the total rank of the equations (22) for accelerations is,

$$R = k - \hat{A}_1 - \hat{A}_2 - \cdots - \hat{A}_N. \tag{27}$$

On the other hand there are a total number of

$$M = \hat{A}_1 + \cdots + \hat{A}_N \tag{28}$$

identities (25) among the L_i.

Another way for the procedure to terminate, is that at the Nth step no new constraint emerges. This means that all the expressions $\lambda_{i_N}^{a_{N+1}} \alpha_{i_N}$ vanish weakly (i.e. up to the previous constraints). So we have $A_{N+1} = \hat{A}_{N+1}$ and $\bar{A}_{N+1} = 0$. In this case the same as (27) and (28) would be deduced from (23) but with N replaced by $(N+1)$.

Summarizing the whole procedure, we see that by processing the Euler–Lagrange equations in a special manner, we finally obtain a total number of R independent equations for accelerations which generally may be less than the total number of degrees of freedom k. On the other hand we can obtain some M relations in the form of (25) among the L_i which hold identically. As we will see in the next section each of these identities corresponds to a GT of the system. Therefore the total number of degrees of freedom k is the sum of gauge degrees of freedom M and the number of independent equations for accelerations R.

One point to be mentioned parenthetically is that the set of Lagrangian constraints which lead to GTs correspond to first-class Hamiltonian constraints, in the Dirac terminology. On the other hand, the set of Lagrangian constraints which give relations to determine a number of undetermined accelerations are related to second-class Hamiltonian constraints. However, the exact inter-relationship is difficult to investigate, but some features can be found in [14].

It should also be noted that the number of dynamical degrees of freedom is still less than R by the number

$$S = \bar{A}_1 + \bar{A}_2 + \cdots + \bar{A}_N$$

which is the number of Lagrangian constraints. That is why, although we find some equations including acceleration by differentiating the constraints, the constraint equations by themselves put stronger restrictions on the dynamical variables. Roughly speaking, S degrees of freedom are either fixed in time or have constant velocities. The same restrictions are also put on the initial values. The situation, however, is different from what is usually implied by the conservation laws. In the latter case some functions of coordinates and velocities do not change with time, but their value can be everything, depending on the initial conditions which can be chosen arbitrarily.

3. Gauge transformations

Let us concentrate on relation (25) and see its consequences. If there exists a set of functions $\phi_{si}(q, \dot{q})$ such that the relation

$$\sum_{s=0}^{n} \frac{\mathrm{d}^s}{\mathrm{d}t^s} (\phi_{si} L_i) = 0 \tag{29}$$

holds identically among the L_i, then the action is invariant under the transformation,

$$\delta q_i = \sum_{s=0}^{n} (-1)^s \frac{\mathrm{d}^s f}{\mathrm{d}t^s} \phi_{si}. \tag{30}$$

Here $f(t)$ is some infinitesimal arbitrary function of time with the only restriction that its first n derivatives (including f itself) vanish at the endpoints.

Let us now evaluate the variation of the Lagrangian under the transformation (30):

$$\begin{aligned}
\delta L &= -L_i\delta q_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\\
&\cong -\sum_{s=0}^{n}(-1)^s\frac{\mathrm{d}^s f}{\mathrm{d}t^s}\phi_{si}L_i\\
&\cong -\phi_{0i}L_i f - \sum_{s=1}^{n}(-1)^{s-1}\frac{\mathrm{d}^{s-1} f}{\mathrm{d}t^{s-1}}\,\frac{\mathrm{d}}{\mathrm{d}t}(\phi_{si}L_i)\\
&\cong \cdots\\
&\cong -\left[\sum_{s=0}^{n}\frac{\mathrm{d}^s}{\mathrm{d}t^s}(\phi_{si}L_i)\right]f\\
&= 0
\end{aligned}$$

where the symbol $\cong$ means equality up to a total time derivative, and in the final line we have used (29). Therefore the variation of action is a combination of derivatives of $f(t)$ at the endpoints, which vanish by assumption. This means that the transformation (30) is really a GT. The above result is valid for any arbitrary trajectory $q_i(t)$ and not necessarily for those who satisfy the equations of motion. The reason is that all the algebraic manipulations of the previous section are some operation on the L_i, without considering $L_i = 0$.

In general, there may exist several relations similar to (29), to be distinguished with the (discrete or continuous) index a:

$$\sum_{s=0}^{n_a}\frac{\mathrm{d}^s}{\mathrm{d}t^s}(\phi_{si}^{(a)}L_i) = 0 \qquad a = 1,\ldots,M. \tag{31}$$

The corresponding GTs are specified by infinitesimal arbitrary functions of time $f_a(t)$. So an arbitrary GT has the form,

$$\delta q_i = \sum_{a=1}^{M}\sum_{s=0}^{n_a}(-1)^s\frac{\mathrm{d}^s f_a}{\mathrm{d}t^s}(\phi_{si}^{(a)}). \tag{32}$$

It seems useful to add some remarks about the generalization of the results to the field theory. Suppose some dynamical system is described by a set of fields $q_i(x,t)$ and a local Lagrangian:

$$L = \int \mathrm{d}x\,\mathcal{L}(q_i(x,t), \partial_x q_i(x,t), \partial_t q_i(x,t)). \tag{33}$$

The equations of motion are

$$L_i(x,t) = \int \mathrm{d}y\,W_{ij}(x,y)\ddot{q}_j(y,t) + \alpha_i(x,t) \qquad i = 1,\ldots,N \tag{34}$$

where N is the number of fields,

$$\alpha_i(x,t) = \int \mathrm{d}y\,\frac{\delta^2 L}{\delta q_j(y,t)\delta\dot{q}_i(x,t)}\dot{q}_j(y,t) - \frac{\delta L}{\delta q_i(x,t)} \tag{35}$$

and

$$W_{ij}(x,y,t) = \frac{\delta^2 L}{\delta\dot{q}_i(x,t)\delta\dot{q}_j(y,t)}. \tag{36}$$

All the derivatives of the Lagrangian are functional derivatives, which for the second derivatives normally lead to expressions including Dirac delta functions and their derivatives.

The Hessian matrix can be viewed as an operator valued $N \times N$ matrix multiplied by $\delta(x-y)$. We consider those singular Lagrangians whose singularity is in the discrete part of the Hessian matrix. The null eigenspace of the Hessian matrix can be spanned by the basis vectors $\lambda_z^{a_1}(x) = \lambda^{a_1}\delta(z-x)$ where λ^{a_1} is some null eigenvector of the discrete part of the Hessian matrix. Multiplication of the equations of motion with $\lambda_z^{a_1}(x)$ yields the primary Lagrangian constraint:

$$\gamma^{a_1}(z,t) = \int \mathrm{d}x\, \lambda_i^{a_1}\delta(z-x)L_i(x,t) = \lambda_i^{a_1}L_i(z,t) = \lambda_i^{a_1}\alpha_i(z,t).$$

If $\lambda_z^{a_1}(x)$ contains derivatives of a delta function it means that one can eliminate acceleration by combining the $L_i(x,t)$ and their spatial derivatives.

The process goes on in the same manner as the previous section. If the system possesses gauge symmetry, one can find relations similar to (29), as follows:

$$\sum_{s=0}^{n_\alpha} \int \mathrm{d}x\, \frac{\partial^s}{\partial t^s}[\phi_{si}^{(\alpha)}(x,z)L_i(x,t)] = 0. \tag{37}$$

Here the spatial variable z and the discrete index α has a role similar to the index a in (31). The GT (32) also takes the form,

$$\delta q_i(x,t) = \sum_{\alpha=1}^{m}\sum_{s=0}^{n_\alpha}(-1)^s \int \mathrm{d}z\, \frac{\partial^s f_\alpha(z,t)}{\partial t^s}\phi_{si}^{(\alpha)}(z,x) \tag{38}$$

where m is the number of arbitrary fields $f_\alpha(z,t)$. Spatial integration in (38) will be removed by delta functions in ϕ. So the GT of the fields $q_i(x)$ will ultimately include spatial and temporal derivatives of the arbitrary fields $f_\alpha(z,t)$.

Sometimes it is easier to find a relation between spatial and temporal derivatives of the $L_i(x,t)$ by direct observation of the equations of motion. If this is the case, it is not difficult to rewrite it in the form of (37) with the use of a Dirac delta function and its derivatives, and then read out the $\phi_{si}^{(\alpha)}(z,x)$ from it directly.

We complete this section by considering a simple example to show how the method works in determining the gauge symmetries of a given system.

Consider the Lagrangian

$$L = \dot{q}_1(\dot{q}_2 + \dot{q}_3) + \dot{q}_2\dot{q}_3 - \dot{q}_3 q_4 - V(q) \tag{39}$$

where

$$V(q) = \tfrac{1}{4}q_4^2 + \tfrac{1}{2}q_4(q_2+q_3). \tag{40}$$

The equations of motion can be written as

$$L_i = W_{ij}\ddot{q}_j + \alpha_i = 0 \qquad i = 1,\ldots,4 \tag{41}$$

where

$$W = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{42}$$

and

$$\alpha = \begin{pmatrix} 0 \\ \frac{1}{2}q_4 \\ -\dot{q}_4 + \frac{1}{2}q_4 \\ \dot{q}_3 + \frac{1}{2}(q_2+q_3+q_4) \end{pmatrix}. \tag{43}$$

The Hessian matrix W has the null eigenvector: $\lambda^1 = (0, 0, 0, 1)$. Multiplying the equations (41) from the left by λ^1 gives the primary Lagrangian constraint

$$\gamma^1 \equiv \lambda_i^1 L_i = \dot{q}_3 + \tfrac{1}{2}(q_2 + q_3 + q_4) \tag{44}$$

which is the same as L_4. Adding $L_5 = \frac{\mathrm{d}}{\mathrm{d}t}\gamma^1(q, \dot{q})$ to the previous L_i, the resulting equations in this step are,

$$L_{i_1} \equiv W^1_{i_1 j}\ddot{q}_j + \alpha^1_{i_1} = 0 \tag{45}$$

where

$$W^1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{46}$$

and

$$\alpha^1 = \begin{pmatrix} 0 \\ \frac{1}{2}q_4 \\ -\dot{q}_4 + \frac{1}{2}q_4 \\ \dot{q}_3 + \frac{1}{2}(q_2 + q_3 + q_4) \\ \frac{1}{2}(\dot{q}_2 + \dot{q}_3 + \dot{q}_4) \end{pmatrix}. \tag{47}$$

Besides λ^1 with one more zero component, W^1 also has the new null eigenvector $\lambda^2 = (1, 1, -1, 0, -2)$. Again multiplication by λ^2 gives the secondary Lagrangian constraint:

$$\gamma^2 \equiv \lambda^2_{i_1} L_{i_1} = -\dot{q}_2 - \dot{q}_3. \tag{48}$$

Adding $L_6 = \mathrm{d}\gamma^2/\mathrm{d}t = 0$ to the previous equations gives

$$L_{i_2} \equiv W^2_{i_2 j}\ddot{q}_j + \alpha^2_{i_2} = 0 \tag{49}$$

where

$$W^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \tag{50}$$

and

$$\alpha^2 = \begin{pmatrix} 0 \\ \frac{1}{2}q_4 \\ -\dot{q}_4 + \frac{1}{2}q_4 \\ \dot{q}_3 + \frac{1}{2}(q_2 + q_3 + q_4) \\ \frac{1}{2}(\dot{q}_2 + \dot{q}_3 + \dot{q}_4) \\ 0 \end{pmatrix}. \tag{51}$$

At this step there exists the new null eigenvector $\lambda^3 = (1, 0, 0, 0, 0, 1)$ for W^2. However, multiplying the equations (49) by λ^3 does not give any new constraint, since the relation $\lambda^3_{i_2} L_{i_2} = 0$ holds identically. Although we did not succeed in increasing the rank of equations for acceleration, we found a relation between the L_i which holds without using the equations of motion:

$$\lambda^3_{i_2} L_{i_2} = L_1 + L_6 = 0. \tag{52}$$

Remembering the definitions of L_6 and L_5, (52) is equivalent to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(-2L_4) + \frac{\mathrm{d}}{\mathrm{d}t}(L_1 + L_2 - L_3) + L_1 = 0 \tag{53}$$

which is in the appropriate form of relation (29). In this problem it is also possible, but difficult, to find (53) by direct inspection of the equations of motion. This shows that in the generic case one cannot rely on the *trial and error* method.

Finally we can find the GT of the system, by using (30), as follows.

$$\begin{cases} \delta q_1 = \dot{f} - f \\ \delta q_2 = \dot{f} \\ \delta q_3 = -\dot{f} \\ \delta q_4 = 2\ddot{f}. \end{cases} \tag{54}$$

The variation of the Lagrangian under the transformation (54) is

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}[-\dot{f}(q_2 + q_3)]$$

which shows the gauge invariance of the action.

4. Schwinger model

The generalized Schwinger model in the bosonized version is described [10] by the Lagrangian:

$$L = \int \mathrm{d}x\, [-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + (e_+\epsilon^{\mu\nu} - e_- g^{\mu\nu})\partial_\nu\phi A_\mu + \tfrac{1}{2}ae^2 A_\mu A^\mu] \tag{55}$$

where e_+ and e_- are related to the coupling constants of the right and left mover fermions with the gauge field A_μ and

$$e^2 = \tfrac{1}{2}(e_+^2 + e_-^2). \tag{56}$$

The undefined parameter a arises in the process of bosonization [11, 10]. The Lagrangian (55) reduces to the Lagrangian of an ordinary Schwinger model by choosing $e_- = 0$ and $a = 0$, and to the axial Schwinger model by choosing $e_+ = 0$ and $a = 2$, both models possessing gauge invariance. For $e_+ = -e_-$ one obtains the chiral Schwinger model. Generalized and chiral Schwinger models are not gauge invariant for any choice of a.

The Lagrangian (55) can also be written explicitly in the form

$$\begin{aligned} L = \int \mathrm{d}x\, [&\tfrac{1}{2}(\dot{\phi}^2 - \phi'^2) + \tfrac{1}{2}(\dot{A}_1^2 + A'^2_0 - 2\dot{A}_1 A'_0) + e_+(A_0\phi' - A_1\dot{\phi}) \\ &-e_-(A_0\dot{\phi} - A_1\phi') + \tfrac{1}{2}ae^2(A_0^2 - A_1^2)] \end{aligned} \tag{57}$$

where dot and prime mean differentiation with respect to time and space respectively.

The equations of motion can be written as:

$$\begin{aligned} L_\phi &\equiv \ddot{\phi} - e_+\dot{A}_1 - e_-\dot{A}_0 - \phi'' + e_+A'_0 + e_-A'_1 = 0 \\ L_{A_0} &\equiv A''_0 - \dot{A}'_1 - e_+\phi' + e_-\dot{\phi} - ae^2A_0 = 0 \\ L_{A_1} &\equiv \ddot{A}_1 - \dot{A}'_0 + e_+\dot{\phi} - e_-\phi' + ae^2A_1 = 0. \end{aligned} \tag{58}$$

The second equation does not include any acceleration and is a constraint. This can be better seen by observing the singularity of Hessian matrix:

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \delta(x - y) \tag{59}$$

where the discrete indices run over the fields ϕ, A_0 and A_1 respectively. The Hessian matrix (59) has the null eigenvector $\lambda^1 = (0, 1, 0)g(z)$, with arbitrary $g(z)$, but as a basis of the null eigenspace we can choose

$$\lambda^1(z) = (0, 1, 0)\delta(z - x). \tag{60}$$

Multiplying the equations of motion (58) from the left with $\lambda^1(z)$ and using (34) one obtains the Lagrangian constraint

$$\gamma^1(z, t) = \int \mathrm{d}x\, \delta(z - x) L_{A_0}(x, t) = L_{A_0}(z, t). \tag{61}$$

Then we can introduce $L_4(x, t) = \frac{\partial}{\partial t}\gamma^1(x, t)$ and add it to equations (58). Using (34) and the field-theoretic counterpart of (14) the result can be shown as

$$W^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ e_- & 0 & -\frac{\partial}{\partial x} \end{pmatrix} \delta(x - y) \tag{62}$$

and

$$\alpha^1 = \begin{pmatrix} -e_+\dot{A}_1 - e_-\dot{A}_0 - \phi'' + e_+A'{}_0 + e_-A'{}_1 \\ A''{}_0 - \dot{A}'_1 - e_+\phi' + e_-\dot{\phi} - ae^2 A_0 \\ -\dot{A}'_0 + e_+\dot{\phi} - e_-\phi' + ae^2 A_1 \\ \dot{A}''_0 - e_+\dot{\phi}' - ae^2\dot{A}_0 \end{pmatrix}. \tag{63}$$

The matrix W^1 has the new null eigenvector $(-e_-, 0, \frac{\partial}{\partial x}, 1)g(x)$ with arbitrary function $g(x)$, but the null eigenspace can be spanned by

$$\lambda^2(z) = (-e_-, 0, \tfrac{\partial}{\partial x}, 1)\delta(z - x). \tag{64}$$

Multiplying by (63) gives

$$\begin{aligned} \gamma^2(z, t) &= \int \mathrm{d}x\, \lambda^2_{i_1} L_{i_1}(x, t) \\ &= \int \mathrm{d}x\, \lambda^2_{i_1} \alpha_{i_1}(x, t) \\ &= (ae^2 - e_-^2)(A'{}_1 - \dot{A}_0) + e_+e_-(\dot{A}_1 - A'{}_0). \end{aligned} \tag{65}$$

This is a secondary Lagrangian constraint. Putting (64) in the first line of the above relations and recalling that $L_4 = \frac{\partial}{\partial t} L_{A_0}$ we see that

$$\gamma^2(x, t) = -e_- L_\phi(x, t) + \tfrac{\partial}{\partial x} L_{A_1}(x, t) + \tfrac{\partial}{\partial t} L_{A_0}(x, t). \tag{66}$$

For the chiral and generalized Schwinger model we should go one step further, and add the time derivative of (65) to the previous equations. Finally we have five equations for acceleration but their rank is three. That is enough to solve them for the dynamics of one of the fields, say ϕ, because we can then use two constraints (61) and (65) in order to find the evolutions of the other two fields. The chiral and generalized Schwinger models possess only one dynamical field and the other two fields have no dynamics.

For the ordinary and axial Schwinger models the situation is completely different. In these two cases it is easy to see that the final expression (65) vanishes, which means gauge invariance of the models. For the ordinary Schwinger model (66) reduces to

$$\frac{\partial}{\partial x} L_{A_1}(x,t) + \frac{\partial}{\partial t} L_{A_0}(x,t) = 0 \tag{67}$$

which by using (65), (64) and (60) (or by direct observation) can be written as,

$$\int \mathrm{d}z \left[-\frac{\partial}{\partial z}\delta(z-x) \right] L_{A_1}(z,t) + \frac{\partial}{\partial t}\left[\int \mathrm{d}z\, \delta(z-x) L_{A_0}(z,t) \right] = 0. \tag{68}$$

In this form (68) is exactly similar to (37) with the identifications,

$$\begin{aligned} \phi_{1,2}(z,x) &= \delta(z-x) \\ \phi_{0,3}(z,x) &= -\frac{\partial}{\partial z}\delta(z-x) \end{aligned} \tag{69}$$

and the remaining $\phi_{si}(z,x)$ as zero. Now using (38), the GT of the fields are as follows:

$$\begin{aligned} \delta\phi &= 0 \\ \delta A_0 &= -\int \mathrm{d}z \frac{\partial}{\partial t} f(z,t)\delta(z-x) = -\frac{\partial}{\partial t} f(x,t) \\ \delta A_1 &= \int \mathrm{d}z\, f(z,t) \left(-\frac{\partial}{\partial z}\delta(z-x) \right) = \frac{\partial}{\partial x} f(x,t) \end{aligned} \tag{70}$$

which can be written in the covariant form as $A_\mu \to A_\mu - \partial_\mu f$ and $\phi \to \phi$.

For the axial Schwinger model, (66) with the use of (60) and (64) can be written as,

$$\int \mathrm{d}z \left[-\frac{\partial}{\partial z}\delta(z-x)\; L_{A_1}(z,t) - e_-\delta(z-x) L_\phi(z,t) \right] + \frac{\partial}{\partial t}\left[\int \mathrm{d}z\, \delta(z-x) L_{A_0}(z,t) \right] = 0 \tag{71}$$

comparing with (38) gives the nonvanishing $\phi_{si}(z,x)$ as follows:

$$\begin{aligned} \phi_{1,2}(z,x) &= \delta(z-x) \\ \phi_{0,1}(z,x) &= -e_-\delta(z-x) \\ \phi_{0,3} &= -\frac{\partial}{\partial z}\delta(z-x). \end{aligned} \tag{72}$$

The resulting GT in this case is

$$\begin{aligned} \delta\phi &= -e_- f(x,t) \\ \delta A_0 &= -\frac{\partial}{\partial t} f(x,t) \\ \delta A_1 &= \frac{\partial}{\partial x} f(x,t). \end{aligned} \tag{73}$$

The transformations (70) and (73) are the well known GTs of the corresponding models.

As can be seen, while the ordinary and axial Schwinger models are gauge invariant, the chiral and generalized Schwinger models, as they stand can by no means possess gauge symmetry. However, it is not possible to bring back, or put in by hand, the gauge symmetry (as is sometimes claimed [12]) in a model which essentially lacks it. Nevertheless it is possible to construct dynamically equivalent models of which some are gauge invariant and some are not.

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Appendix. A-type constraints

A set of independent functions of velocity and coordinates are not necessarily independent constraints. That is why, one may eliminate velocities and find a number of constraints between coordinates only. These are recognized as A-type constraints in the terminology of [9]. Time derivatives of A-type constraints may have vanishing combinations with the remaining B-type constraints.

For example, the constraints $\gamma_1 = \dot{q}_1 + q_3$, $\gamma_2 = \dot{q}_1 - q_2$ and $\gamma_3 = \dot{q}_2 + \dot{q}_3$ are independent functions of coordinates and velocities, but in fact they are equivalent to the two constraints $\gamma_1' = \gamma_1$ and $\gamma_2' = \gamma_1 - \gamma_2$, and the identity $\gamma_3 = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1 - \gamma_2)$.

Any set of N constraints $\gamma_n(q, \dot{q})$ are, in principle, equivalent to N_A A-type constraints $\gamma_{n_A}(q)$, N_B B-type constraints $\gamma_{n_B}(q, \dot{q})$ and a set of $\tilde{N}$ identities containing first-order time derivatives of constraints. After some algebraic manipulations the evolving A-type and B-type constraints should satisfy the following conditions.

(1) A-type constraints should have maximal rank N_A, i.e.

$$\mathrm{rank}\left(\frac{\partial \gamma_{n_A}}{\partial q_i}\right) = N_A. \tag{74}$$

(2) B-type constraints should be N_B independent functions of velocities, such that

$$\mathrm{rank}\left(\frac{\partial \gamma_{n_B}}{\partial \dot{q}_i}\right) = N_B. \tag{75}$$

(3) As functions of velocities, time derivatives of A-type constraints should be independent of B-type ones. Since $\frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}\gamma_{n_A}}{\mathrm{d}t}\right) = \frac{\partial \gamma_{n_A}}{\partial q_i}$, this condition can be written as,

$$\mathrm{rank}\begin{pmatrix} \frac{\partial \gamma_{n_A}}{\partial q_i} \\ \frac{\partial \gamma_{n_B}}{\partial \dot{q}_i} \end{pmatrix} = N_A + N_B. \tag{76}$$

Applying the above procedure to the set of constraints $\gamma^{\bar{a}_n}(q, \dot{q})$, we see that two changes are necessary in the nth step of section 2. First, a number of additional identities among the L_i should be considered since, as is mentioned in the text, the $\gamma^{\bar{a}_n}$ are some combinations of the L_i. Then the same number should be subtracted from the number of constraints. Second, in order to write new equations for accelerations, one should consider first derivatives of B-type and second derivatives of A-type constraints. Condition (76) then ensures that the added equations for acceleration are independent of each other. So the process goes on with no change except that a recombination of constraints should be carried out.

References

[1] Gracia X and Pons J M 1988 *Ann. Phys.* **187** 355
[2] Gomis J, Henneaux M and Pons J M 1990 *Class. Quantum Grav.* **7** 1089
[3] Henneaux M, Teitelboim C and Zanelli J 1990 *Nucl. Phys.* B **332** 169
[4] Sundermeyer K 1982 *Constrained Dynamics* (Berlin: Springer)

[5] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University Press)
[6] Castellani L 1982 *Ann. Phys.* **143** 357
[7] Costa M E V, Girotti H O and Simoes T J M 1985 *Phys. Rev.* D **32** 405
[8] Cawley R 1979 *Phys. Rev. Lett.* **42** 413
[9] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics, A Modern Perspective* (New York: Wiley-Interscience)
[10] Yan-Gang Miao, Jian-Ge Zhou and Yao-Yang Liu 1993 *J. Phys. G: Nucl. Part. Phys.* **19** 1797
[11] Rajaraman R 1985 *Phys. Lett.* B **154** 305
[12] Mitra P and Rajaraman R 1989 *Phys. Lett.* B **225** 276
[13] Girotti H O, Rothe H J and Rothe K D 1986 *Phys. Rev.* D **33** 514
[14] Batlle C, Gomis J, Pons J M and Roman-Roy N 1986 *J. Math. Phys.* **27** 2953