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Full and partial gauge fixing

A. Shirzad[a)]
Institute for Studies in Theoretical Physics and Mathematics, P. O. Box 5531, Tehran 19395, Iran and Department of Physics, Isfahan University of Technology, Isfahan, Iran

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Gauge fixing may be done in different ways. We show that using the chain structure to describe a constrained system enables us to use either a full gauge, in which all gauged degrees of freedom are determined, or a partial gauge, in which some first class constraints remain as subsidiary conditions to be imposed on the solutions of the equations of motion. We also show that the number of constants of motion depends on the level in a constraint chain in which the gauge fixing condition is imposed. The relativistic point particle, electromagnetism, and the Polyakov string are discussed as examples and full or partial gauges are distinguished. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2709846]

I. INTRODUCTION

There are two well-known methods to construct the constraint structure of a constrained system.[1,2] First, the *level by level* method,[3] in which at any level the consistency equations are solved simultaneously to find the constraints of the next level. Second, the *chain by chain* method,[4] in which the consistency of every constraint produces the next constraint in a given chain. In the second method the constraints are organized in separate first and second class chains. As is well known, the first class constraints (FCC's) are generators of gauge transformations which correspond to the emergence of arbitrary functions of time in the solutions of equations of motion. The relationship between the first class constraints, the generating function of gauge transformations, and the arbitrary functions of time has intensively been studied in the literature.[6–10] However, this relationship can be better understood in the context of the chain by chain method. Every first class constraint chain of N entries corresponds, in the solutions of equations of motion, to an arbitrary function of time together with its derivatives up to the level $(N-1)$.

In the presence of gauge transformations any physical state corresponds to an orbit in the phase space, i.e., the gauge orbit, along which only the arbitrary functions of time change. Gauge transformations, generated by FCC's, just translate the system along the gauge orbits, without changing the physical state. Gauge fixing means that coordinates describing the gauge orbits are determined such that a one to one correspondence exists between the physical states of the system and the points of the remaining subspace of the phase space. In this way it is needed to impose extra constraints by hand on the system to fix the gauges. We call these constraints "the gauge fixing conditions" (GFC's).

Suppose for simplicity that we have chosen some suitable coordinates in which the FCC's are converted to some momenta. Then the gauge fixing is equivalent to determining the time behavior of the corresponding conjugate coordinates. In other words, the GFC's should have nonvanishing Poisson brackets, at least, with a subset of FCC's. When the gauge is fixed, arbitrary functions of time (or arbitrary fields in the case of a field theory) are no more present in the solutions of the equations of motion.

Depending on the way the GFC's are chosen, some FCC's may still remain as additional conditions which should be imposed on the physical solutions of the problem. This feature, though

[a)]Electronic mail: shirzad@ipm.ir

encountered, for instance, in the string theory, has not been recognized so far in the context of the constrained systems. One effect of this feature concerns the number of initial constants appearing in the solutions of the equations of motion, which will be discussed in this paper.

In the following section we will first review the basic concepts of the chain structure as well as the proposal of Ref. 5 which implies *full gauge fixing* of a given gauge theory. In this article we show that this is only one possibility. In fact, it is also possible to fix the gauge *partially*. Analyzing a simple toy model in Sec. II we will explain the main idea of this paper. That is, the details of the gauge fixing (including the number of initial conditions) depend on the *gauge fixing level* defined as the definite level of constraint chains on which the GFC's are imposed.

Section III is devoted to investigating the relativistic point particle. The constraint structure, the gauge transformation (which in this case is reparametrization) and the corresponding generating function, the full and partial gauges, and finally the number of initial constants in this problem are discussed. The next interesting model is electromagnetism, which is studied in Sec. IV mostly in relationship with the problem of gauge fixing. We will show that the famous Coulomb gauge is a full gauge, while the Lorentz gauge is of a completely different nature which we call a *primitive gauge*.

In Sec. V we will investigate the constraint structure of the bosonic string theory and analyze different gauges traditionally used in the literature. We show that the famous covariant gauge is a partial one which implies imposing the Virosoro constraints as subsidiary conditions on the solutions of the equations of motion as well as on the physical states in the quantum theory. On the other hand, the light cone gauge, although disturbing the manifest covariance of the theory, is a full gauge which preserves only purely physical degrees of freedom. In Sec. VI we will give our concluding remarks.

II. GAUGE FIXING IN CHAIN STRUCTURE

Suppose, for simplicity, that we have just one primary constraint, ϕ_1, in a system described by the canonical Hamiltonian H_c. The dynamics of every function $g(q,p)$ is achieved by

$$\dot{g}(q,p) = \{g, H_t\}, \tag{2.1}$$

where H_t is the total Hamiltonian given as

$$H_t = H_c + \lambda \phi_1, \tag{2.2}$$

in which λ is the undetermined Lagrange multiplier. Following the conventional consistency procedure of Dirac,[1] i.e., $\dot{\phi} \approx 0$ (where $\approx$ means weak equality), the second level constraint emerges as

$$\phi_2 = \{\phi_1, H_c\} \approx 0. \tag{2.3}$$

We are interested in the first class systems where $\{\phi_2, \phi_1\} \approx 0$. Therefore, the consistency of ϕ_2, from Eq. (2.1), gives $\phi_3 = \{\phi_2, H_c\}$ and so on. In this way a constraint chain is derived via *the chain rule*

$$\phi_{n+1} = \{\phi_n, H_c\}, \tag{2.4}$$

provided that $\{\phi_n, \phi_1\}$ vanish (at least weakly) so that the Lagrange multiplier λ is not determined at any stage. A first class chain terminates at level N, say, where

$$\{\phi_N, H_c\} \approx 0. \tag{2.5}$$

Hence, the corresponding Lagrange multiplier remains undetermined. It can be shown that the solutions of the equations of motion in this case contain one arbitrary function of time and its time derivatives up to order $N-1$.[2] For instance, in the simple case where $\{\phi_N, H_c\}$, as well as $\{\phi_n, \phi_1\}$ for all n, vanish strongly, it is shown[6] that the gauge transformations are generated by the following function:

$$G = \sum_{s=1}^{N} (-1)^s \phi_s(q,p) \frac{\mathrm{d}^{N-s}\eta(t)}{\mathrm{d}t^{N-s}}, \tag{2.6}$$

where $\eta(t)$ is an infinitesimal arbitrary function of time.

The above considerations can be easily generalized to a multichain system. For this reason one should only add a chain label to the constraints as well as the arbitrary functions of time. The number of chains, arbitrary functions of time, and the primary first class constraints are the same. However, the chains may have different lengths. It should be noticed that for a generic system it is not an easy task to arrange the constraints as chains. In fact, this requires a special algorithm to be followed as given in Ref. 4.

Now let us see how the gauge can be fixed. Since all the first class constraints ϕ_n are generators of gauge transformation, it may seem that one should impose the same number of GFC's as that of constraints, i.e., the GFC $\omega_n=0$ should be imposed to fix the gauge transformation generated by ϕ_n. However, the key point is that the GFC's should remain valid in the course of time in the same way as the FCC's themselves. This fact brings our attention to two points: first, if the GFC's are not chosen appropriately, their consistency may lead to extra constraints which may *overdetermine* the system; second, one may shorten the way through finding all GFC's needed to fix the gauge by giving a smaller number of GFC's and finding the rest of them by following their consistency conditions. This is, in fact, the main idea of Ref. 5, where the authors proposed imposing the *primary* GFC $\omega_N \approx 0$, where ω_N is conjugate to the terminating element of the chain (while commuting with the others), i.e.,

$$\{\omega_N, \phi_n\} = \chi(q,p)\delta_{n,N}, \tag{2.7}$$

where χ should not vanish on the surface of the constraints.

To get a better idea of how this method works suppose that the terminating element is one of the momenta, say, p_k. Clearly the conjugate coordinate q_k is not contained in the previous constraints (otherwise we would not have a first class system). Then from Eq. (2.6) the gauge transformation of q_k is just $\delta q_k = \eta(t)$. In other words q_k is an arbitrary function depending on the gauge. Once this function is chosen by the gauge $\omega_N = q_k - f(t) \approx 0$, where $f(t)$ is some given function of time, the gauge would be fixed completely. Since ω_N is an explicit function of the time its consistency leads to the next GFC via the formula

$$\dot{\omega}_N \equiv \omega_{N-1} = \{\omega_N, H_c\} + \frac{\partial \omega_N}{\partial t}. \tag{2.8}$$

Using the chain rule [Eq. (2.4)] and the Jacobi identity one can show that ω_{N-1} is conjugate to ϕ_{N-1}, and so on. Hence, the GFC's in turn obey the chain rule

$$\omega_{n-1} = \{\omega_n, H_c\} + \frac{\partial^{N-n+1} f}{\partial t^{N-n+1}} \tag{2.9}$$

and constitute conjugate pairs with FCC's,

$$\{\omega_n, \phi_n\} = (-1)^{N-n}\chi. \tag{2.10}$$

The procedure goes on up to the last step where consistency of ω_1 determines the Lagrange multiplier as

$$\lambda = \frac{(-1)^N\{\omega_1, H_c\} + \partial^N f/\partial t^{N+1}}{\chi(q,p)}. \tag{2.11}$$

The above procedure, which we call *full gauge fixing*, leads to a complete fixing of the gauge. The reduced phase space achieved by imposing the whole FCC's and GFC's has the dimension of $2K-2N$, where $2K$ is the dimension of the original phase space. Therefore, the number of physical degrees of freedom (which come through second order differential equations of motion) would be

$K-N$. In this way, in full gauge fixing, the number of constants to be determined by the initial conditions is $2(K-N)$. For a multichain system this would be clearly $2(K-\Sigma_a N_a)$, where a is the chain index.

Now let us see what happens if the gauge fixing does not begin from the terminating element of the chain. We call such a method as *partial gauge fixing*. Suppose, for some reason, one has begun fixing the gauge from some intermediate element in the chain, say, from ϕ_M, where $M<N$. By this we mean that one imposes the GFC ω_M, instead of ω_N, such that

$$\{\omega_M,\phi_n\}=\chi(q,p)\delta_{n,M}. \tag{2.12}$$

Note specially that ω_M commutes with the constraints succeeding ϕ_M as well as the ones preceding it. Then the consistency process gives the set of GFC's $\omega_{M-1},\omega_{M-2}\ldots$, similar to full gauge fixing. At the last step λ is determined similar to Eq. (2.11) with N replaced by M. In this way the set $\phi_1,\ldots,\phi_M,\omega_M,\ldots,\omega_1$ serves as a system of second class constraints which leads to a reduced phase space with dimension of $2K-2M$.

However, we are left with the constraints $\phi_{M+1},\ldots,\phi_N$, which are not yet fixed during the gauge fixing process. Although the gauge is fixed so that there remains no arbitrary function of time in the solutions of equations of motion, *one should still impose the remaining constraints* $\phi_{M+1},\ldots,\phi_N$ *on the solutions* to get a consistent physical system. In other words, the classical solutions are achieved by solving second order differential equations for $K-M$ variables together with imposing $N-M$ constraints (appearing in the shape of first or zeroth order differential equations in configuration space). Therefore, the number of constants to be determined by initial conditions is $2K-2M-(N-M)=2K-N-M$.

The partial gauge fixing has also considerable effects on the quantization procedure. We remind that there are two methods for quantizing a first class system. The first one is to fix the gauges completely and then quantize the reduced phase space variables by converting them to operators and converting their Dirac brackets to commutators. The second method is to quantize all the original phase space variables by converting the original Poisson brackets to commutators and then specializing the physical states by imposing the condition

$$\mathrm{FCC}|\mathrm{phys}\rangle=0. \tag{2.13}$$

The reason for this condition is that the generator of gauge transformations in the general case can be written in terms of first class constraints. Hence, Eq. (2.13) results in the physical condition $G|\mathrm{phys}\rangle=0$ for any set of arbitrary functions.

The quantization procedure in a partial gauge fixed system is a mixture of both methods. In this case, the variables of the $2K-2M$ dimensional reduced phase space should first be quantized by converting the following Dirac brackets to commutators:

$$\{f,g\}_{DB}=\{f,g\}-\{f,\psi_r\}C^{rs}\{\psi_s,g\}, \tag{2.14}$$

where

$$\psi_r,\psi_s\epsilon\{\phi_1,\ldots,\phi_M,\omega_M,\ldots,\omega_1\}, \tag{2.15}$$

and C^{rs} is the inverse of

$$C_{rs}=\{\psi_r,\psi_s\}. \tag{2.16}$$

Then the following condition should be imposed on states to achieve the physical ones,

$$\hat{\phi}_n|\mathrm{phys}\rangle=0,\quad M+1<n<N, \tag{2.17}$$

where $\hat{\phi}_n$ is the operator version of the constraint ϕ_n.

To see the above ideas more clearly consider a simple toy model with (x,y,z) as the variables, described by the Lagrangian

$$L = \dot{x}\dot{y} - yz. \tag{2.18}$$

The momentum p_z emerges as the primary constraint. The total and canonical Hamiltonian read

$$\begin{aligned} H_t &= H_c + \lambda p_z, \\ H_c &= p_x p_y + yz. \end{aligned} \tag{2.19}$$

Using the chain rule [Eq. (2.4)], the following first class constraint chain is derived:

$$\begin{aligned} \phi_1 &= p_z, \\ \phi_2 &= -y, \\ \phi_3 &= -p_x. \end{aligned} \tag{2.20}$$

Since the last element of the chain commutes strongly with H_c, the generator of gauge transformation can be written from Eq. (2.6) as

$$G = -p_z\ddot{\eta} - y\dot{\eta} + p_x\eta. \tag{2.21}$$

Suppose that we want to fix the gauge fully. This is done by imposing the GFC

$$\omega_3 = x - f(t). \tag{2.22}$$

Using Eq. (2.9), the consistency of ω_3 gives the next two GFC's as

$$\begin{aligned} \omega_2 &= p_y - \dot{f}(t), \\ \omega_1 &= -z - \ddot{f}(t). \end{aligned} \tag{2.23}$$

Consistency of ω_1, using the total Hamiltonian [Eq. (2.19)], determines the Lagrange multiplier as

$$\lambda = -\frac{d^3 f}{dt^3}. \tag{2.24}$$

As can be seen, this system with three degrees of freedom obeys three first class constraints, which means that the system is completely gauged (has no further dynamical degree of freedom). So in a full gauge there remains no dynamics in the system. In other words, all the variables are determined by the choice of the function $f(t)$. Moreover, since $N=K=3$, the number of initial constants is zero.

Now let us fix the gauge partially in this system. Suppose that one prefers to fix the gauge by imposing the GFC

$$\omega_1' = z - g(t), \tag{2.25}$$

which fixes the value of z whose variation is generated by the FCC p_z. Consistency of ω_1', using Eq. (2.8), determines the Lagrange multiplier as

$$\lambda = \dot{g}(t). \tag{2.26}$$

The total Hamiltonian turns out to be

$$H_t = p_x p_y + g(t)y + \dot{g}(t)p_z. \tag{2.27}$$

The four dimensional reduced phase space acquires the following equations of motion:

$$\dot{x} = p_y, \quad \dot{p}_y = -g(t),$$
$$y = 0, \quad p_x = 0. \tag{2.28}$$

Equations in the first line are derived from the total Hamiltonian [Eq. (2.27)], while the ones in the second line are the constraints that remained at the tail of the constraint chain [Eq. (2.20)] without imposing corresponding GFC's. In this example these two sets of equations are separated; this point is not essential for the general case. The final solution of the equations of motion for the remaining two degrees of freedom, i.e., x and y, are obtained as

$$\ddot{x} = -g(t), \quad y = 0. \tag{2.29}$$

Integrating the first equation brings in two constants, in agreement with the formula $2K-N-M=6-3-1=2$.

The above considerations can be seen more clearly in the Lagrangian framework. The Euler-Lagrange equations of motion due to the Lagrangian [Eq. (2.18)] read

$$\frac{\delta L}{\delta x} = \ddot{y} = 0,$$
$$\frac{\delta L}{\delta y} = \ddot{x} + z = 0, \tag{2.30}$$
$$\frac{\delta L}{\delta z} = y = 0.$$

The first equation can result from the third one, which requires that y is fixed at zero. The remaining equation constrains the time behavior of x and z. If one determines x as the given function $f(t)$, then z would be completely determined as $\ddot{f}(t)$. Conversely, if one determines z as a definite function $g(t)$, then x should be found by integrating $g(t)$ twice, which brings in two constants of integration.

Finally, let us take a look at the problem of quantization of the model. In the full gauge fixing the reduced phase space is null and no degree of freedom remained to be quantized. On the other hand, in the partial gauge fixing the canonical operators $(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y)$ describe a quantum particle in two dimensions. However, the physical subspace due to the conditions [Eq. (2.17)] is restricted to the states satisfying

$$\hat{y}|\psi\rangle = \hat{p}_x|\psi\rangle = 0. \tag{2.31}$$

These two conditions are so powerful to kill all the states except the single state $|\psi\rangle = |y=0, p_x=0\rangle$ with the wave function

$$\psi(x,y) = \frac{1}{2\pi}\delta(y). \tag{2.32}$$

III. RELATIVISTIC POINT PARTICLE

Consider a relativistic point particle in a D dimensional Minkovski space-time described by the action

$$S = \frac{1}{2}\int d\tau(\eta^{-1}\dot{X}^\mu \dot{X}_\mu - m^2\eta), \tag{3.1}$$

where m is the mass of the particle, "dot" means differentiating with respect to τ, the proper time, and $\eta(\tau)$ is an auxiliary variable called the *ein-bin* variable. The canonical momenta conjugate to X^μ and η are, respectively,

$$P_\mu = \eta^{-1}\dot{X}_\mu, \quad P_\eta = 0. \tag{3.2}$$

So P_η is the primary constraint. The canonical and total Hamiltonians are as follows:

$$\begin{aligned} H_c &= \frac{1}{2}(\eta P^\mu P_\mu + m^2\eta), \\ H_t &= H_c + \lambda P_\eta. \end{aligned} \tag{3.3}$$

The consistency process gives the following constraint chain:

$$\begin{aligned} \phi_1 &= P_\eta, \\ \phi_2 &= -\frac{1}{2}(P^\mu P_\mu + m^2). \end{aligned} \tag{3.4}$$

Since $\{\phi_2, H_t\}$ vanishes strongly, the generator of gauge transformation, using Eq. (2.6), can be written in terms of an arbitrary infinitesimal function $\epsilon(t)$ as

$$G = -\dot{\epsilon}\phi_1 + \epsilon\phi_2. \tag{3.5}$$

Let us see which transformation is generated by G. Using Eqs. (3.4) and (3.5), the variations of X^μ and η under the action of G are, respectively,

$$\delta X^\mu \equiv \{X^\mu, G\} = \epsilon P^\mu, \tag{3.6}$$

$$\delta\eta \equiv \{\eta, G\} = \dot{\epsilon}. \tag{3.7}$$

Using the definition of P_μ, Eq. (3.6) gives

$$\delta X^\mu = \epsilon\eta^{-1}\dot{X}^\mu. \tag{3.8}$$

Equation (3.7) shows that $\eta(\tau)$ is somehow arbitrary. Therefore, assuming $\xi(\tau) \equiv -\epsilon\eta^{-1}$, we have

$$\delta X^\mu = -\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\xi(\tau), \tag{3.9}$$

$$\delta\eta = -\dot{\eta}\xi - \eta\dot{\xi}. \tag{3.10}$$

It is easily seen that the action [Eq. (3.1)] is invariant under the reparametrization

$$\tau \to \tau' = \tau + \xi(\tau), \tag{3.11}$$

provided that the transformed variables behave as follows:

$$X'^\mu(\tau') = X^\mu(\tau), \tag{3.12}$$

$$\eta'(\tau')\mathrm{d}\tau' = \eta(\tau)\mathrm{d}\tau. \tag{3.13}$$

Now we show that the variations derived in Eqs. (3.9) and (3.10) correspond to an infinitesimal reparametrization. To do this, using Eq. (3.12) we can write

$$\delta X^\mu \equiv X'^\mu(\tau) - X^\mu(\tau) \cong X'^\mu(\tau) - X'^\mu(\tau') \cong -\frac{\partial X'^\mu}{\partial \tau}\delta\tau \cong -\frac{\partial X^\mu}{\partial \tau}\xi(\tau),$$

where $\cong$ means equality up to the first order quantities in terms of the infinitesimal variables. On the other hand, Eqs. (3.13) and (3.11) imply that

$$\eta'(\tau')(1+\dot{\xi}) = \eta(\tau),$$

which gives

$$\delta\eta \equiv \eta'(\tau) - \eta(\tau) \cong \eta'(\tau') - \eta(\tau') \cong \eta(\tau) - \dot{\xi}(\tau)\eta(\tau) - \eta(\tau') \cong -\frac{\partial \eta}{\partial \tau}\mathrm{d}\tau - \dot{\xi}(\tau)\eta(\tau)$$
$$\cong -\dot{\eta}(\tau)\xi(\tau) - \dot{\xi}(\tau)\eta(\tau).$$

These calculations show that the gauge generating function G in Eq. (3.5) is, in fact, the generator of infinitesimal reparametrizations.

Now let us proceed to the problem of gauge fixing. It is clear that at most one arbitrary function of time would appear in the solutions of the equations of motion. Therefore, different gauges correspond to the choice of the variable which is determined by the gauge (e.g., one of the X^μ's or η). One simple choice is considering η as the given function $f(t)$ by imposing the GFC $\omega_1' = \eta - f(t)$. Since this gauge fixes only the first entry in the constraint chain [Eq. (3.4)], it is a partial gauge. The consistency of ω_1' from Eq. (2.8) determines the Lagrange multiplier λ in Eq. (3.3) as $\lambda = \dot{f}$. The canonical equations of motion read

$$\dot{X}^\mu = \eta P^\mu,$$
$$P^\mu = 0. \tag{3.14}$$

Equations (3.14) together with the remaining constraint $\phi_2 = P_\mu P^\mu + m^2 = 0$ determine all the variables. It is easy to see that these equations bring in $2D-1$ constants of integration, in agreement with the formula $2K - N - M = 2(D+1) - 2 - 1 = 2D - 1$.

It is worth to note the Lagrangian equations of motion, i.e.,

$$\frac{\delta L}{\delta \eta} = -\eta^{-2}\dot{X}^\mu \dot{X}_\mu - m^2 = 0, \tag{3.15}$$

$$\frac{\delta L}{\delta X^\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\eta^{-1}\dot{X}_\mu) = 0. \tag{3.16}$$

Equation (3.15) is acceleration-free and serves as a first level (as well as last level) Lagrangian constraint.[14] It is easily seen that the Lagrangian equations of motion [Eqs. (3.15) and (3.16)] are not sufficient to determine all the variables. However, assuming $\eta(t)$ as an arbitrary function, we can determine X^μ's in terms of $\eta(t)$ and the constants P^μ by integrating the equation $P^\mu = \eta^{-1}\dot{X}^\mu$. The number of independent P^μ's is $D-1$ according to the condition $P^\mu P_\mu + m^2 = 0$ resulting from Eq. (3.15). Integrating $\dot{X}^\mu = \eta(t)P^\mu$ brings in D further integration constants, adding up to $2D-1$, as expected.

One can also consider a full gauge by imposing a desired time dependence for one of the X^μ's or a combination of them. The most famous gauge is the temporal one, in which X^0 is assumed to be the same as the proper time. The primary GFC in this gauge is

$$\omega_2 = X^0 - \tau. \tag{3.17}$$

Using Eq. (2.8) the consistency of ω_2 gives the next GFC as

$$\omega_1 = \eta P^0 - 1. \tag{3.18}$$

The set of canonical equations [Eq. (3.14)] together with the constraints ϕ_1 and ϕ_2 and the GFC's ω_2 and ω_1 determines all the variables as

$$\eta = \frac{1}{P^0},$$
$$X^0 = \tau, \tag{3.19}$$
$$X^i = \frac{P^i}{P^0}\tau + x^{0i}.$$

The total number of constants in this case is $2(D-1)$, where $D-1$ of them are the independent P^μ's (remember that the constraint ϕ_2 implies $P^{0^2}=\Sigma P^{i^2}+m^2$) and $D-1$ of them are the x^{0i}'s. This is in agreement with the formula $2K-2N=2(D+1)-4=2D-2$.

Similar treatment can be done in the light cone coordinates, where $X^\pm \equiv (X^0 \pm X^1)/\sqrt{2}$ and $X^i \equiv X^\mu$, $\mu=2,\ldots,D$. A full gauge fixing in these coordinates can be achieved by imposing the GFC $\omega_2=X^+-\tau=0$, whose consistency gives $\omega_1=-\eta P_- - 1=0$. The reduced phase space is achieved by omitting the canonical pairs (η, P_η) and (X^+, P_+). Independent variables X^i and X^- can be solved in terms of $2(D-1)$ constants P_i, P_-, X^{0i}, and X^{0-} as

$$X^i(\tau) = -\frac{P_i}{P_-}\tau + X^{0i},$$
$$X^-(\tau) = -\frac{\sum P_i^2 + m^2}{P_-^2}\tau + X^{0-}. \tag{3.20}$$

IV. ELECTROMAGNETISM

Consider the famous action of the electromagnetism as

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}, \tag{4.1}$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. The canonical momenta are $\Pi_\mu = -F_{0\mu}$ which yield $\phi_1=\Pi_0$ as the primary constraint. The total Hamiltonian reads

$$H_t = H_c + \int d^3x \lambda(x)\Pi_0(x), \tag{4.2}$$

where H_c is the canonical Hamiltonian

$$H_c = \int d^3x\left[\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}F_{ij}F_{ij} - A_0\partial_i\Pi_i\right]. \tag{4.3}$$

We assume the metric to be $g_{\mu\nu}=\mathrm{diag}(-+++)$. Consistency of ϕ_1 gives the secondary constraint $\phi_2=\partial_i\Pi_i$. Consistency of ϕ_2 is fulfilled identically. So we have a constraint chain with two elements.

A full gauge fixing can be achieved by imposing $\omega_2=\partial_i A^i$ as the primary GFC which is conjugate to ϕ_2. Consistency of ω_2 gives the next GFC as $\omega_1=\partial_i\Pi_i+\partial_i\partial_i A_0$, which is weakly equivalent to $\partial_i\partial_i A_0$. Finally, consistency of ω_1 determines λ as any function with vanishing divergence. A well defined Dirac bracket would emerge from the second class set given by ϕ_1, ϕ_2, ω_2, and ω_1, which is well known in the literature.[11] This gauge is the famous Coulomb gauge.

One can also perform a partial gauge by imposing the GFC $\omega_1'=A^0$, which is conjugate to ϕ_1. This gauge determines the Lagrange multiplier $\lambda(x)$ to be identical to zero, which yields $H_t=H_c$. Even though this choice of gauge kills the arbitrariness of the theory, we still remained with the not yet fixed constraint ϕ_2. The canonical equations of motion read

$$\dot{A}_i = \Pi_i + \partial_i A_0, \tag{4.4}$$

$$\dot{\Pi}_i = \partial_j F_{ji}. \tag{4.5}$$

Eliminating the canonical pair (A^0,Π_0) determines the rest of the variables via $\dot{A}_i=\Pi_i$ and Eq. (4.5). These are the same equations that can be derived from the canonical Hamiltonian by eliminating the last term in Eq. (4.3). However, one should note that the resulting equation,

$$\ddot{A}_i = \nabla^2 A_i - \partial_i(\nabla \cdot \mathbf{A}), \tag{4.6}$$

should be considered together with the constraint

$$\partial_i \Pi_i = \partial_i \dot{A}_i = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = 0. \tag{4.7}$$

In other words, the final answer for $\mathbf{A}$ is any solution of the dynamical equation [Eq. (4.6)] with static divergence.

As far as the number of initial constants (in this case initial fields) is concerned, the dynamical equation [Eq. (4.6)] brings in six initial conditions. However, the constraint [Eq. (4.7)] decreases it to five, in agreement with the previous counting formula.

It is worth noting that in the Lagrangian formulation the equations of motion read

$$L^\nu \equiv \partial_\mu F^{\mu\nu} = 0. \tag{4.8}$$

It is clear that the *Eulerian derivatives* L^ν are not independent functions, since $\partial_\nu L^\nu=0$. Therefore, the equations of motion at most can be used to determine three independent fields A_i out of four. However, in the gauge $A^0=0$, Eq. (4.8) for $\nu=i$ gives the dynamical equation [Eq. (4.6)], while for $\nu=0$ the constraint $\partial/\partial t(\nabla\cdot\mathbf{A})=0$ is obtained. One can check that the consistency of this Lagrangian constraint is fulfilled identically according to the equations of motion.

We conclude this section by a discussion on the Lorentz gauge, given in the Lagrangian form as

$$\partial_\mu A^\mu = \partial_0 A^0 + \partial_i A^i = 0. \tag{4.9}$$

Note that the velocity $\dot{A}^0$ cannot be obtained in terms of the phase space variables. On the other hand, using Eq. (4.2) we have

$$\dot{A}^0 = \{A^0, H_t\} = \lambda. \tag{4.10}$$

Hence, in the Hamiltonian framework the Lorentz gauge can be achieved by imposing the GFC

$$\omega^{(\lambda)} = \lambda + \nabla \cdot \mathbf{A} = 0, \tag{4.11}$$

which depends on the Lagrange multiplier as well. This gauge is, in fact, equivalent to choosing the Lagrange multiplier in terms of the physical variables from the very beginning. The dynamics of the system is then given by the total Hamiltonian

$$H_t = \int d^3x \left[\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}F_{ij}F_{ij} - A_0\partial_i\Pi_i - \Pi_0\partial_i A_i \right], \tag{4.12}$$

together with the constraints ϕ_1 and ϕ_2. In this way the canonical equations of motion reproduce Eqs. (4.4) and (4.5) as well as the gauge condition [Eq. (4.9)] which finally yield the wave equations for all A^μ, as expected. It is worth noting that the consistency of $\omega_2^{(\lambda)}$, using Eqs. (2.8) and (4.12), gives

$$\dot{\omega}^{(\lambda)} = \dot{\lambda} + \nabla^2 A^0, \tag{4.13}$$

which from Eq. (4.10) results in the equations of motion for A^0 (i.e., the wave equation). Therefore the consistency of $\omega_2^{(\lambda)}$ is fulfilled identically.

Now this question may arise: "which kind of gauge is the Lorentz gauge, full or partial?" Remember that in the case of a partial gauge, if the primary GFC is conjugate to the Mth level constraint, then after M steps of investigating the consistency of GFC's one would be able to determine the Lagrange multiplier. Furthermore, $N-M$ remaining constraints should be imposed on the solutions of the equations of motion. However, in the case of Lorentz gauge there is no need to follow the consistency process to find the Lagrange multiplier. On the other hand, all the constraints are needed to be imposed on the solutions of the equations of motion. In other words, $M=0$ for Lorentz gauge. So, roughly speaking, we can say that such a gauge is *a primitive gauge*. In other words, all of the four fields A^μ are taken into account within the dynamical equations of motion (i.e., wave equation) and none of them, or no combination of them, is omitted according to the gauge.

More accurately, in the case of a primitive gauge, the meaning of GFC's as additional constraints which reduce the "constraint surface" into the "reduced phase space" should be revised. In such systems the gauge orbits disappear by determining the Lagrange multiplier, rather than by cutting the gauge orbits via imposing the GFC's. The most interesting fact is that, although the gauge is fixed, the original Poisson bracket is unchanged. In other words, no Dirac bracket is needed to describe the algebraic structure of the physical phase space. Especially, in order to quantize the theory, all of the eight fields A^0, A^i, Π_0, and Π_i, should be converted to canonical operators, while the physical subspace of the system is composed of states destroyed by the first class constraints Π_0 and $\partial_i\Pi_i$. However, this quantized system differs from that obtained by quantizing the first class system (without gauge fixation) in the sense that in this case a well defined Hamiltonian, i.e., the quantized version of Eq. (4.12), is responsible for the evolution of the system.

As the final point note that since $M=0$ in the case of a primitive gauge, the number of initial conditions is $2K-N$ for such a system. For electromagnetism in Lorentz gauge the canonical equations due to the Hamiltonian [Eq. (4.12)] bring in eight initial constants where two of them are redundant according to the constraint equations $\Pi_0=0$ and $\partial_i\Pi_i=0$.

V. POLYAKOV STRING

The Polyakov string is introduced[12,13] by the action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu, \tag{5.1}$$

where g_{ab} is the world-sheet metric, g is minus the determinant and g^{ab} is the inverse of g_{ab}, X_μ, $\mu=0,1,\ldots,D-1$ are bosonic fields, and $1/2\pi\alpha'$ is the tension of the string which can be taken to be unity. Since $\dot{g}_{ab}(\equiv\partial_\tau g_{ab})$ is absent from the Lagrangian, the conjugate momentum fields π^{00}, $\pi^{01}(=\pi^{10})$, and π^{11} are primary constraints, i.e.,

$$\pi^{ab} \equiv \frac{\partial L}{\partial \dot{g}_{ab}} \approx 0. \tag{5.2}$$

The remaining momenta and the canonical Hamiltonian read

$$P_\mu \equiv \frac{\partial L}{\partial \dot{X}^\mu} = \sqrt{g}(g^{00}\dot{X}_\mu + g^{01}X'_\mu), \tag{5.3}$$

$$H = \frac{1}{2}\int \mathrm{d}\sigma\left[\frac{1}{g^{00}}\left(\frac{P_\mu}{\sqrt{g}} - g^{01}X'_\mu\right)^2 - g^{11}X'^2\right] = \frac{1}{2}\int \mathrm{d}\sigma\left[\frac{1}{\sqrt{g}g^{00}}(P^2 + X'^2) - \frac{2g^{01}}{g^{00}}P\cdot X'\right], \tag{5.4}$$

where "dot" and "prime" represent differentiating with respect to τ and σ, respectively. Then one should investigate the consistency of the primary constraints π^{ab}. Since the canonical Hamiltonian depends on the string variables (X and P) only through the functions $(P^2+X'^2)$ and $P\cdot X'$, the consistency procedure will give some functions of the metric variables (g_{ab} or g^{ab}) times the above functions, which, as we will see in the following, have weakly vanishing Poisson brackets with each other and with the Hamiltonian. Therefore, without going through detailed calculations, one can guess that there exists no further constraint after the second level.

These observations suggest a change of variables from the original metric components to some suitable combinations of them. Equation (5.4) shows that the following variables are adequate:

$$N_1 = \frac{1}{\sqrt{g}g^{00}}, \tag{5.5}$$

$$N_2 = -\frac{g^{01}}{g^{00}}, \tag{5.6}$$

$$N_3 = \sqrt{g}. \tag{5.7}$$

The variable N_3 is also dictated by the fact that the action is independent of the scale of the metric which can be given by $\sqrt{g}$. The metric can be written in terms of N_i as

$$g_{ab} = \begin{pmatrix} N_3N_1[1-(N_2/N_1)^2] & -N_3N_2/N_1 \\ -N_3N_2/N_1 & -N_3/N_1 \end{pmatrix}. \tag{5.8}$$

Writing the action in terms of N_i, it is clear that their conjugate momenta Π^i are primary constraints. Then using the total Hamiltonian

$$H_t = H + \int \mathrm{d}\sigma \sum_{i=1}^{3} \lambda_i \Pi^i, \tag{5.9}$$

where

$$H = \frac{1}{2}\int \mathrm{d}\sigma[N_1(P^2 + X'^2) + N_2(2P\cdot X')], \tag{5.10}$$

the consistency of Π^3 is satisfied trivially, while the consistencies of Π^1 and Π^2 give, respectively, the following secondary constraints

$$\Phi_1 = \tfrac{1}{2}(P^2 + X'^2),$$
$$\Phi_2 = P \cdot X'. \tag{5.11}$$

The Poisson brackets of the secondary constraints read

$$\begin{aligned}\{\Phi_1(\sigma,\tau),\Phi_1(\sigma',\tau)\} &= \partial\delta(\sigma-\sigma')[P(\sigma',\tau)\cdot X'(\sigma,\tau) + P(\sigma,\tau)\cdot X'(\sigma',\tau)]\{\Phi_1(\sigma,\tau),\Phi_2(\sigma',\tau)\} \\ &= \partial\delta(\sigma-\sigma')[X'(\sigma',\tau)\cdot X'(\sigma,\tau) + P(\sigma,\tau)\cdot P(\sigma',\tau)]\{\Phi_2(\sigma,\tau),\Phi_2(\sigma',\tau)\} \\ &= \partial\delta(\sigma-\sigma')[P(\sigma',\tau)\cdot X'(\sigma,\tau) + P(\sigma,\tau)\cdot X'(\sigma',\tau)],\end{aligned} \tag{5.12}$$

which vanish on the constraint surface. Hence, no other constraint emerges. Using the language of the chain by chain method, we have derived the following three first class constraint chains:

$$\Pi^1, \Pi^2, \Pi^3,$$
$$\Phi_1, \Phi_2. \tag{5.13}$$

The third chain in Eq. (5.13) contains only one element, π^3, i.e., the generator of gauge variation of N_3 which only changes the scale of the world-sheet metric. This is the well-known Weyl symmetry of the Polyakov string. The remaining constraints in Eq. (5.13), i.e., $(\Pi^1, \Pi^2; \Phi_1, \Phi_2)$, generate the reparametrizations in a more complicated way, which is not of our interest here.

Let us investigate the problem of gauge fixing in the Polyakov string. As we observed, all three independent components of the world-sheet metric are gauge variables and can be determined by fixing the gauge. For example, one may assume the world-sheet metric to be flat, i.e., $g_{ab} = \eta_{ab}$, corresponding to $N_1 = -1$, $N_2 = 0$, and $N_3 = 1$. This choice of gauge kills the reparametrization, as well as the Weyl invariance of the Polyakov action. However, since the above CFC's are conjugate to the first level constraints Π^i, the corresponding gauge is a partial one. Therefore, the second level constraints Φ^1 and Φ^2 remain unfixed and should be imposed after all on the solutions of the classical equations on motion. Moreover, in the quantized theory the effect of the Virosoro constraints Φ^1 and Φ^2 on the physical states should vanish. Quantization of the bosonic string in this gauge is known to the string theorists as "the old covariant quantization."[13]

Suppose one had considered that, from the very beginning, the world-sheet metric is flat. Then one would have a different theory without any first class constraint. In such a theory one encounters a traceless energy-momentum tensor, but there will be no justification for imposing the Virosoro constraints Φ^1 and Φ^2 which imply vanishing of the energy-momentum tensor. It is essential to distinguish between the gauge fixed theory of the bosonic string coupled to two dimensional gravity (i.e., Polyakov string) and the theory of bosonic string living on a flat world sheet.

Although the general covariance of the world sheet disappears in the above partial gauge, this gauge has the advantage of keeping the Lorentz covariance of the fields in the target space. This is in contrast with the famous light cone gauge, in which the Lorentz covariance of the coordinate fields X^μ, as well as the general covariance of the world sheet, are destroyed. The light cone gauge is in some sense a full gauge, since as we will see in the following, the assumed GFC's together with their consistency conditions give as many GFC's as the constraints. This gauge is introduced, in terms of the variables N_1, N_2, N_3, $X^\pm \equiv (X^0 \pm X^1)/\sqrt{2}$, and X^i, $i = 2, \ldots, D$, by the following GFC's:

$$\Omega_1 = N_1 + 1, \tag{5.14}$$

$$\Omega_2 = N_2', \tag{5.15}$$

$$\Omega_3 = N_3 - 1, \tag{5.16}$$

$$\omega_2 = X^+ - a\tau - b. \tag{5.17}$$

These are four GFC's, while we have five FCC's. Therefore one more GFC is needed to fix the gauge completely. Consistency of Ω_3 in Eq. (5.16) determines λ_3 in Eq. (5.9) to be zero and fixes the Weyl gauge transformation generated by Π^3. To fix the reparametrizations, let us first consider the consistency of ω_2 in Eq. (5.17). Using Eqs. (2.8) and (5.14)–(5.17) we have

$$\dot{\omega}_2 \equiv \omega_1 = P_- - a. \tag{5.18}$$

This GFC completes the set of required GFC's to fix all the gauge transformations. The consistency of ω_1 gives $(N_1 X'_-)' + (N_2 P_-)'$, which vanishes identically by imposing Eqs. (5.14)–(5.17) and gives no further result. Finally, the consistency of Ω_1 and Ω_2 determines the remaining Lagrange multipliers as $\lambda_1 = \lambda'_2 = 0$. Imposing the boundary conditions determines the constant value of λ_2 to be zero.

The coordinate X^+ as well as the momentum P_- are determined according to GFC's ω_2 and ω_1. Note that, using the constraints Φ_1 and Φ_2, the conjugate fields P_+ and X^- may be determined in terms of the transverse coordinates and momenta (i.e., X^i and P_i) as follows:

$$P_+ \approx \frac{1}{2a}(P^i P_i + X^{i\prime} X_i{}'), \tag{5.19}$$

$$X^{-\prime} \approx \frac{1}{a}(P_i X^{i\prime}). \tag{5.20}$$

In this way all the gauges are fixed and there remain only transverse coordinates as the physical fields which possess independent dynamics in the classical level. This result may be compared with the light cone gauge in the case of relativistic point particle in which X^- is remained as a dynamical coordinate [see Eq. (3.20)]. To quantize the theory one should find the Dirac brackets due to these ten constraints (i.e., five FCC's and five GFC's given above). It is not difficult to see that

$$\{X^i(\sigma,\tau), P_j(\sigma',\tau)\}_{DB} = \delta^i_j \delta(\sigma - \sigma'), \tag{5.21}$$

while all other Dirac brackets vanish. Therefore the system may be easily quantized (after imposing suitable boundary conditions) by quantizing just the transverse coordinates with no need to impose subsidiary conditions on the physical states.

VI. CONCLUDING REMARKS

We showed, in this paper, that the chain by chain method in constructing the constraint structure of a gauge theory provides a suitable framework for classifying different types of the gauges. From this point of view we introduced full and partial gauges.

The full gauge fixing is achieved when the gauge fixing conditions are chosen to be conjugate to the last elements of the first class chains. In this category of gauges, the consistency of primary gauge fixing conditions produces newer ones. Repeating this procedure leads to the emergence of enough number of gauge fixing conditions such that every gauge generator (i.e., first class constraint) has its conjugate among the set of assumed and produced gauge fixing conditions. Therefore, the gauge would be fixed completely, so that none of the generators of the gauge transformations remained unfixed. The relativistic point particle in temporal and light cone gauges, electromagnetism in Coulomb gauge, and the Polyakov string in light cone gauge are shown to be examples of full gauges.

The partial gauge fixings concern cases in which the primary gauge fixing conditions are proposed to be conjugate to some first class intermediate constraints in the constraint chains which we call the gauge fixing level. The consistency of primary gauge fixing conditions produce newer ones which are conjugate to the constraints preceding the gauge fixing level. Therefore, the constraints succeeding this level remain unfixed. Hence, it is necessary to take into account the remaining unfixed constraints as subsidiary conditions which should be imposed, at the classical level, on the solutions of the equations of motion, and should kill, at the quantum level, the physical states. In other words, imposing a partial gauge on the original Hamiltonian or Lagrangian is not enough; it is also necessary to follow up the history of the constraint structure of the system and impose the original constraints on the solutions of the gauge fixed system. Relativistic point particle in the gauge which determines the ein-bin variable, electromagnetism in the vanishing potential gauge ($A^0=0$), and the Polyakov string in the old covariant gauge are examples of partial gauges.

An interesting observation in studying electromagnetism is that the Lorentz gauge has a special character which we call a primitive gauge. In this system, one fixes the gauge by determining the Lagrange multipliers from the very beginning. Therefore, although the gauge freedom is fixed directly, all the first class constraints remained unfixed and should be considered as subsidiary conditions.

We also had a discussion on the number of initial constants in different gauges. We showed that this number is the smallest in the case of a full gauge.

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[14] We remind that the nth level Lagrangian constraint corresponds to $(n-1)$th level Hamiltonian one Ref. 3.