Classification of constraints
using chain by chain method

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Abstract

We introduce "chain by chain" method for constructing the constraint structure of a system possessing both first and second class constraints. We show that the whole constraints can be classified into completely irreducible first or second class chains. We found appropriate redefinition of second class constraints to obtain a symplectic algebra among them.

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1 Introduction

Constrained systems, are known from almost 1950 [1, 2]. These systems are in fact the basis of gauge theories. Modern formalisms of quantization, such as BRST [3] and BV [4], are constructed on constrained systems. For a singular Lagrangian system, primary constraints emerge since the momenta $p_i = \partial L / \partial \dot{q}_i$ are not independent functions of $(q, \dot{q})$. However, if we do not care about the origin of primary constraints, we can suppose that we are given a canonical Hamiltonian H_c, together with a number of primary constraints ϕ_1^a, $a = 1, \ldots, m$.

The equation of motion for an arbitrary function $g(q, p)$ reads [1]:

$$\dot{g} = \{g, H_T\}, \tag{1}$$

where the total Hamiltonian H_T is defined as

$$H_T = H_c + v_a \phi_1^a, \tag{2}$$

in which v_a's are Lagrange multipliers (LM). Since the constraints should vanish at any arbitrary time, their consistency from (1) requires that

$$\dot{\phi}_1^a = \{\phi_1^a, H_T\} \approx 0. \tag{3}$$

Secondary constraints may emerge from consistency of primary constraints. Consistency of any constraint χ implies that $\{\chi, H_T\} \approx 0$, where the weak equality $\approx$ means equality on the constraint surface. This may lead to one of the following cases [5, 6]:

i) One of the Lagrange multipliers is determined. Roughly speaking, this occurs when $\{\chi, \phi_1^a\} \napprox 0$ for some a.

ii) Consistency is achieved identically, i.e. $\{\chi, H_c\} \approx 0$ and $\{\chi, \phi_1^a\} \approx 0$.

iii) A new constraint emerge. This is the case when $\{\chi, \phi_1^a\} \approx 0$ but $\{\chi, H_c\} \napprox o$.

In cases (i) and (ii) the consistency process should stop, but in case (3) one should go on through the consistency of new constraint $\tilde{\chi} = \{\chi, H_c\}$ and so on. This procedure leads to a *constraint structure*, which is studied in references such as [5, 7, 11]

We assume for simplicity that:

a) No ineffective constraint (such as $\phi = x^2$) emerges.

b) "Bifurcation" does not happen when investigating the consistency conditions, i.e. equations like $xp \approx 0$ do not emerge.

Moreover, we say two functions (in phase space) "commute", when they have vanishing Poisson bracket.

Starting with a system of primary constraints ϕ_1^a, one can use different methods to find out all of the constraints. The method which is well known in the literature [5, 7, 8, 9, 10] is the *level by level* method. In this method, at the n'th level of consistency, say, one tries to solve the equations $\{\phi_n^a, H_T\} \approx 0$ to find the maximum possible number of LM's. This depends on the rank of matrix $\{\phi_n^a, \phi_1^b\}$. In principle, ϕ_n^a's divide into two subclasses: $\phi_n^{a_1}$'s which commute with ϕ_1^a's and $\phi_n^{a_2}$'s which do not commute. The latter i.e. second class constraints, lead to determining a number of LM's. The former lead to constraints of the next level, which their number is less than or equal to $\phi_n^{a_1}$'s (depending on the number of independent non vanishing functions among $\{\phi_n^{a_1}, H_c\}$). As we will discuss in the last section, this method and the resulting *constraint algebra* is very complicated and a number of its mathematical statements should begin with "in principle it is possible that ...".

We propose an alternative procedure to produce the constraints of the system. Our strategy is as follows:

1) We investigate the consistency of constraints *chain by chain*, i.e. when a new constraint emerge we go on by considering its consistency. In this way, beginning with a primary constraint, one produces a constraint chain. When a chain terminates, we consider the next primary constraint and *knit* its corresponding chain and so on. (In some cases we should go back and continue to knit a chain which had been considered as a terminated one. Such cases will be discussed later.)

2) All of the equalities should be considered as weak equalities. By weak equality, at each step, we mean equality on the surface of constraints known up to that step. When a new constraint is recognized, it should be added to the set of known constraints. If needed, one can redefine any function, such as a constraint or canonical Hamiltonian, by adding a combination of known constraints to it.

In this way, the whole system of constraints is managed in a chain structure. Moreover, the set of constraints is irreducible by construction. The details are given in sections 2-4. In sections 2 and 3, we explain the method for systems with one or two primary constraints respectively. In section 4 we extend the method to an arbitrary system with several primary constraints. Section 5 is devoted to some examples.

The advantages of this method and comparison with the existing level by level method is given in section 6 which is the concluding section.

2　One Chain System

Consider a system with one PC, say ϕ_1. The total Hamiltonian is

$$H_T = H_c + v\phi_1. \tag{4}$$

Following the strategy given in section 1, there is only one chain, whose constraints obey the recursion relation

$$\phi_n = \{\phi_{n-1}, H_c\}. \tag{5}$$

Suppose first that the chain terminates at level N according to case (i) of the previous section, i.e.

$$\{\phi_N, \phi_1\} \approx \eta(q, p) \not\approx 0 \tag{6}$$

To investigate the constraint algebra, we arm ourselves with the following lemma.

Lemma 1:

For the chain described by (5,6)

$$\begin{aligned}
\{\phi_j, \phi_i\} &\approx 0 & i + j \leq N \tag{7} \\
\{\phi_{N-i}, \phi_{i+1}\} &\approx (-1)^i \eta & i = 0, \ldots, N - 1 \tag{8}
\end{aligned}$$

proof: Suppose $i < j$. It is obvious that $\{\phi_j, \phi_1\} \approx 0$ for $j < N$, since otherwise the chain would be terminated before level N. Assuming

$$\{\phi_j, \phi_i\} \approx 0, \qquad j = 1, \ldots, N - i \tag{9}$$

we prove that

$$\{\phi_{j+1}, \phi_i\} \approx 0, \qquad j = 1, \ldots, N - i - 1. \tag{10}$$

Using (5) and Jacobi identity, we have

$$\{\phi_j, \phi_{i+1}\} = \{\{\phi_j, \phi_i\}, H_c\} - \{\{\phi_j, H_c\}, \phi_i\} \tag{11}$$

From (9), $\{\phi_j, \phi_i\}$ is a combination of ϕ_k's with $k < N$. Therefore the first term on RHS of (11) vanishes weakly. The second term is just $\{\phi_{j+1}, \phi_i\}$ which vanishes according to the assumption (9). In this way (7) is proved inductively. The second part of the lemma can be proved by inserting $j = N - i$ in (11) which gives

$$\{\phi_{N-i}, \phi_{i+1}\} \approx -\{\phi_{N-i+1}, \phi_i\} \approx \ldots \approx (-1)^i \{\phi_N, \phi_1\} \tag{12}$$
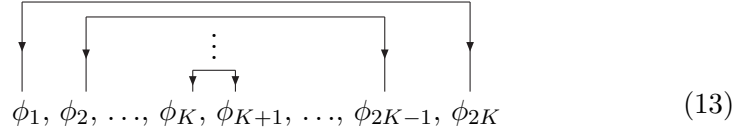
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Then (8) follows from (6).

Corollaries:

a) All of the constraints of the chain are second class.

b) N is even. (Since in (8) one should necessarily have $N - i \neq i + 1$.)

c) The following diagram shows schematically that each entry in the second half of the chain $(\phi_1, \ldots, \phi_K, \phi_{K+1}, \ldots, \phi_{2K})$ is *conjugate* to (have non vanishing Poisson bracket with) its partner in the first half.

$$\phi_1, \ \phi_2, \ \ldots, \ \phi_K, \ \phi_{K+1}, \ \ldots, \ \phi_{2K-1}, \ \phi_{2K} \tag{13}$$

We call a chain with the above properties a *self-conjugate* (SCC) second class chain. However, using appropriate redefinitions, one can replace the chain with an equivalent set $(\Omega_1, \ldots, \Omega_K, \Omega_{K+1}, \ldots, \Omega_N)$ obeying the symplectic algebra:

$$\delta_{ij} = \{\Omega_i, \Omega_j\} \approx J_{ij} \tag{14}$$

where J is the symplectic $2K \times 2K$ matrix:

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} \tag{15}$$

The proof is given in appendix A.

The relation (14) is the best thing that one can find for the algebra of a set of second class constraints. In fact, since $\Delta^{-1} = -J$, one can easily define the Dirac brackets and get into the reduced phase space.

The other possibility for the chain to terminate at level N is $\dot{\phi}_N = \{\phi_N, H_T\} \approx 0$ or equivalently
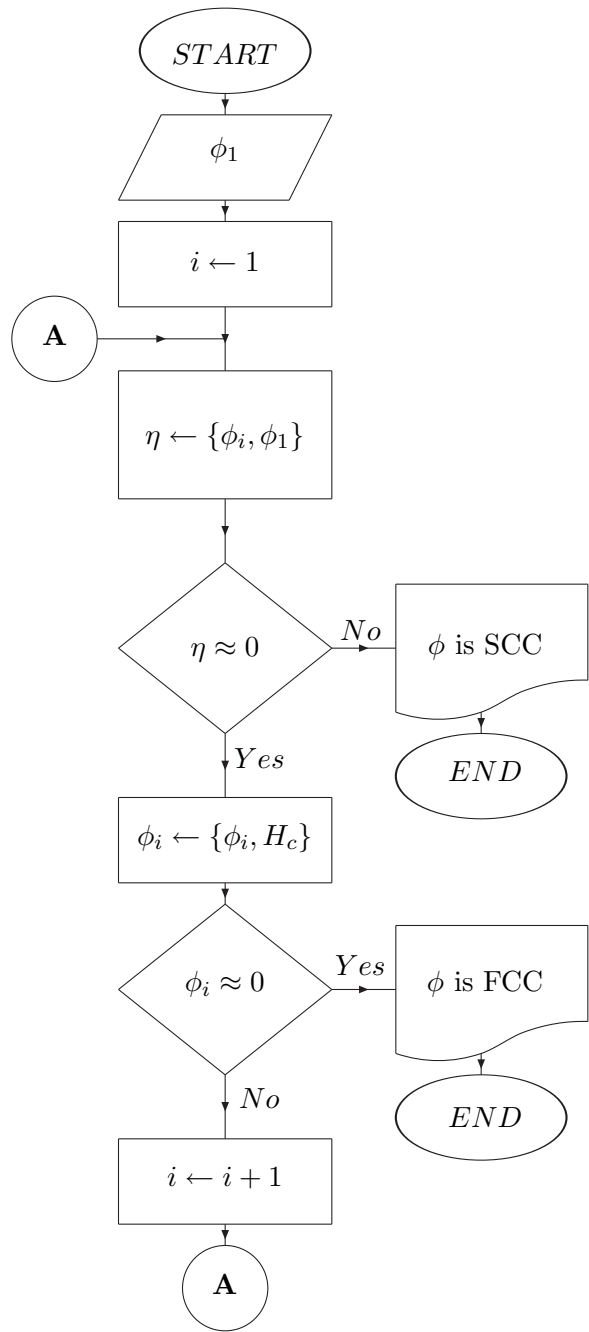
$$\{\phi_N, \phi_1\} \approx 0 \tag{16}$$
$$\{\phi_N, H_c\} \approx 0 \tag{17}$$

Following the steps of the previous lemma one finds that

$$\{\phi_i, \phi_j\} \approx 0 \qquad i, j = 1, \ldots, N. \tag{18}$$

In other words all of the constraints of the chain are first class. We name such a chain a first class chain (FCC).

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Concluding, we found that *a one-chain system is either completely first class or completely second class.* In the former case the Lagrange multiplier remains undetermined as an arbitrary function of time ; but in the latter case it would be determined and using the Dirac brackets, one can get into the reduced phase space. The following flowchart shows the whole procedure schematically.

START

ϕ_1

$i \leftarrow 1$

A

$\eta \leftarrow \{\phi_i, \phi_1\}$

$\eta \approx 0$ —No→ ϕ is SCC

END

Yes

$\phi_i \leftarrow \{\phi_i, H_c\}$

$\phi_i \approx 0$ —Yes→ ϕ is FCC

END

No

$i \leftarrow i + 1$

A

3 Two chain system

Suppose we are given a system with two PC's, say ϕ_1 and ψ_1. The total Hamiltonian is

$$H_T = H_c + v\phi_1 + w\psi_1. \tag{19}$$

Following our strategy, we first go through the consistency of ϕ_1 and knit the ϕ-chain via the relation
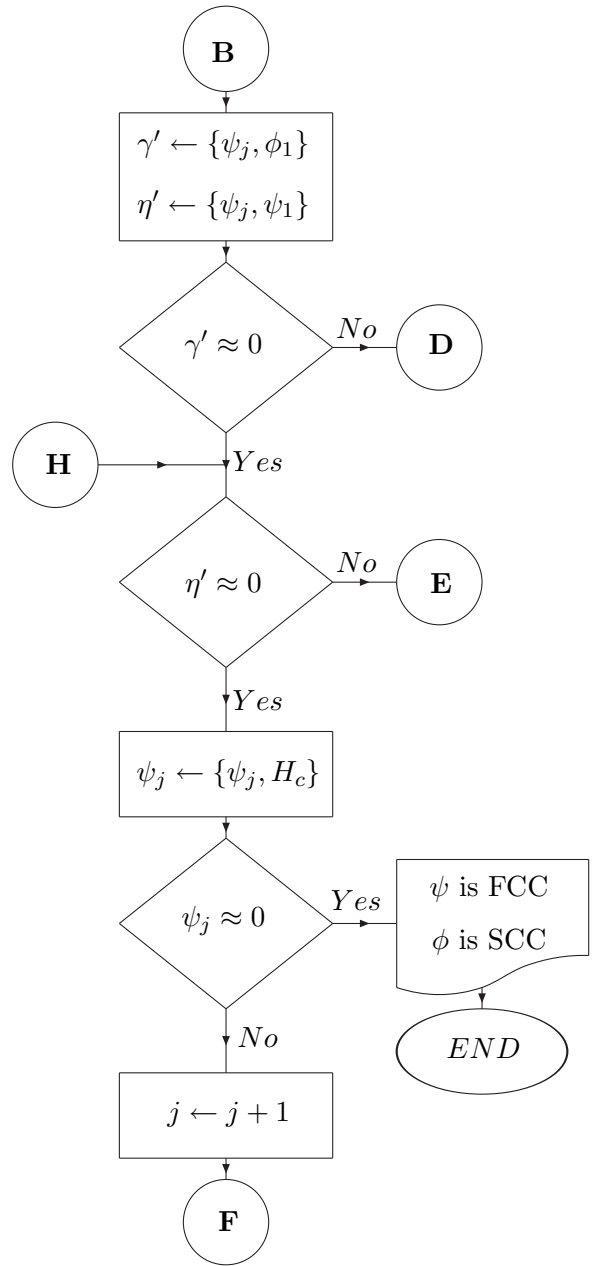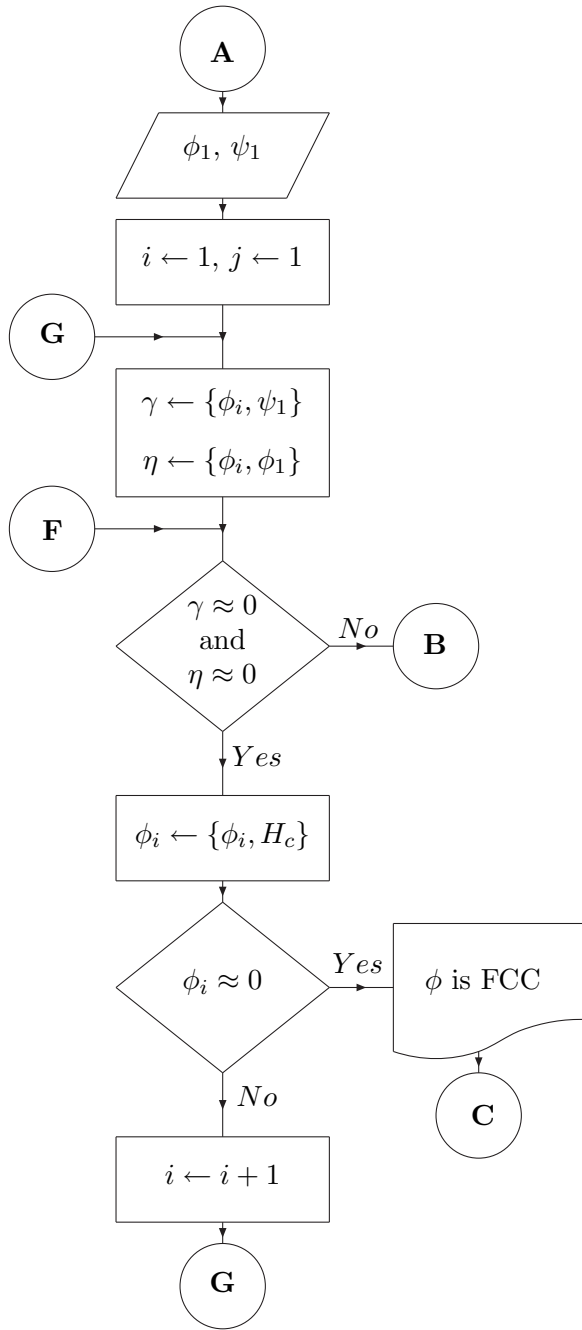
$$\phi_{i+1} = \{\phi_i, H_c\}. \tag{20}$$
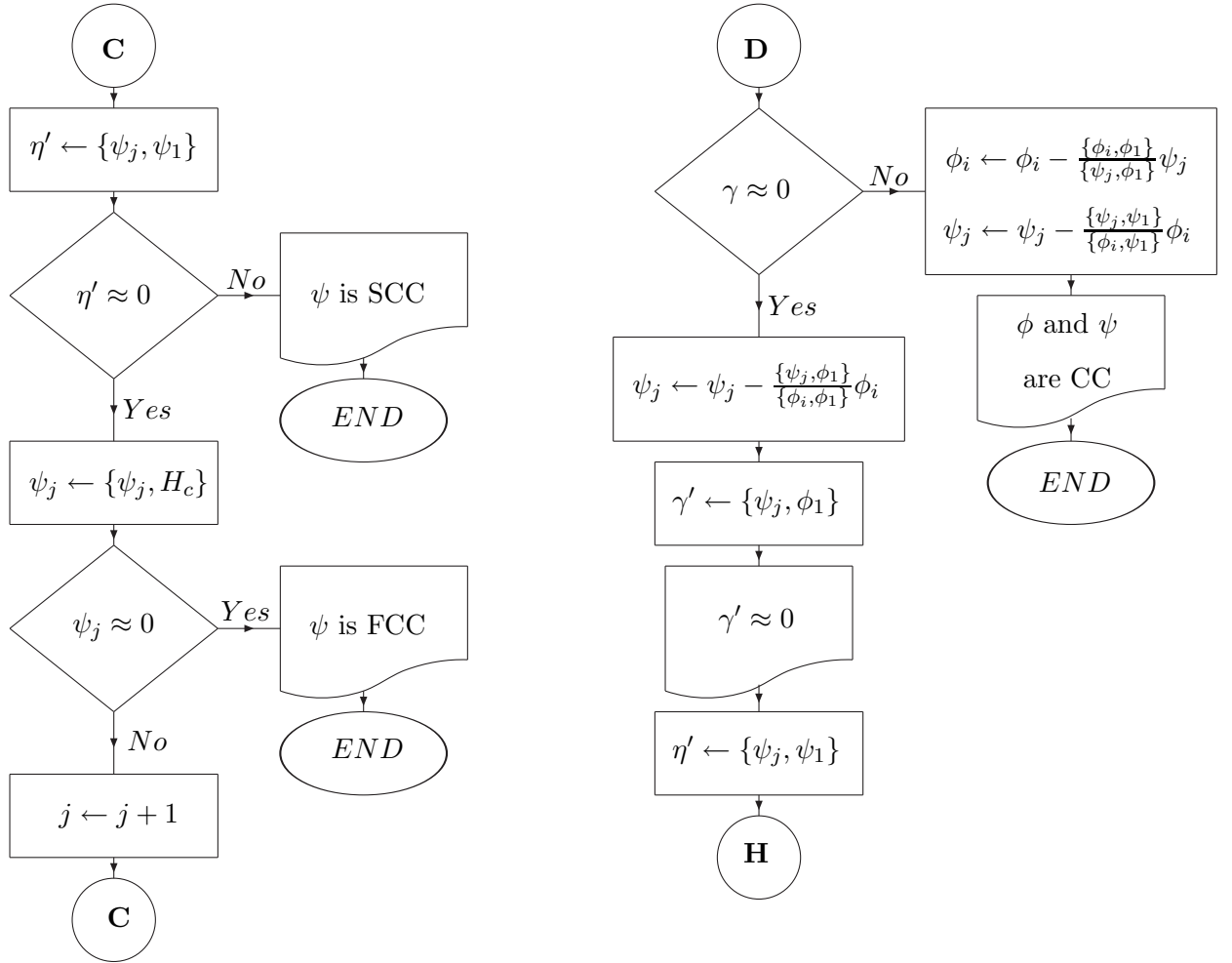
Then we knit the ψ-chain in the same way:

$$\psi_{i+1} = \{\psi_i, H_c\}. \tag{21}$$

Suppose ϕ_N and ψ_M are the last members of the corresponding chains. We define

$$
\begin{aligned}
\eta &= \{\phi_N, \phi_1\} \\
\gamma &= \{\phi_N, \psi_1\} \\
\gamma' &= \{\psi_M, \phi_1\} \\
\eta' &= \{\psi_M, \psi_1\}.
\end{aligned} \tag{22}
$$

Our program for constructing the system of constraints are summarized in the flowcharts A to E, given below. The function of each part of the program are then briefly explained. Lemmas 2-4, which are needed to verify the results will come afterward.

C

$\eta' \leftarrow \{\psi_j, \psi_1\}$

$\eta' \approx 0$ — No → ψ is SCC → *END*

Yes

$\psi_j \leftarrow \{\psi_j, H_c\}$

$\psi_j \approx 0$ — Yes → ψ is FCC → *END*

No

$j \leftarrow j + 1$

C

D

$\gamma \approx 0$ — No → $\phi_i \leftarrow \phi_i - \frac{\{\phi_i, \phi_1\}}{\{\psi_j, \phi_1\}}\psi_j$

$\psi_j \leftarrow \psi_j - \frac{\{\psi_j, \psi_1\}}{\{\phi_i, \psi_1\}}\phi_i$

→ ϕ and ψ are CC → *END*

Yes

$\psi_j \leftarrow \psi_j - \frac{\{\psi_j, \phi_1\}}{\{\phi_i, \phi_1\}}\phi_i$

$\gamma' \leftarrow \{\psi_j, \phi_1\}$

$\gamma' \approx 0$

$\eta' \leftarrow \{\psi_j, \psi_1\}$

H
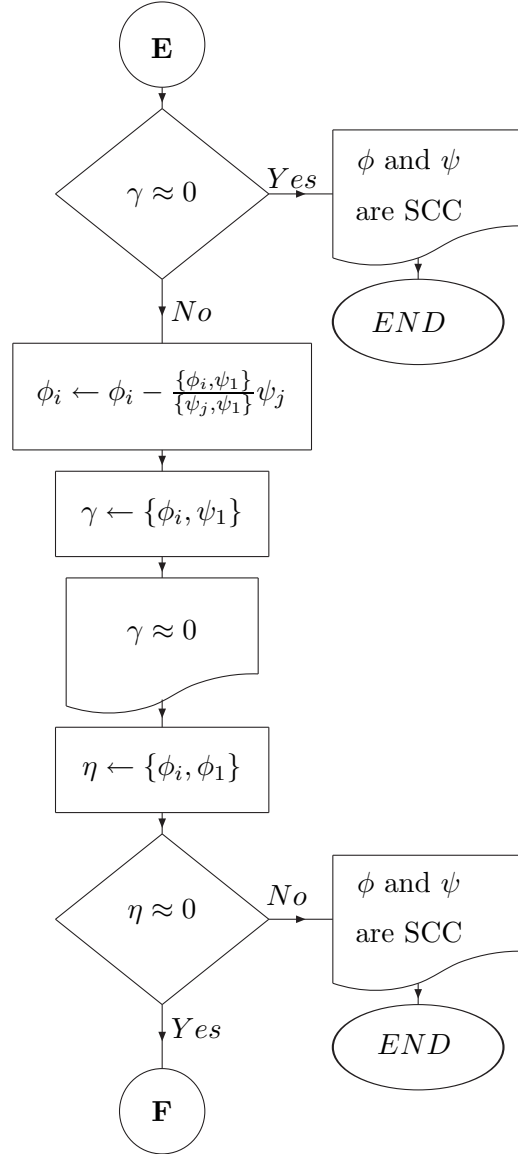
A- As is apparent from flowchart, this part of program corresponds to knitting the ϕ-chain. The constraint ϕ_i may be the terminating element of the chain if $\{\phi_i, \phi_1\}$ and/or $\{\phi_i, \psi_1\}$ does not vanish (case i in section 1). If this is the case, one should go to the sub-program **B** to knit the ψ-chain. If $\{\psi_i, \phi_1\}$ and $\{\phi_i, \psi_1\}$ both vanish, one should consider $\{\phi_i, H_c\}$. The chain terminates if $\{\phi_i, H_c\} \approx 0$. Lemma-3 then shows that the ϕ-chain is first

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class and commute with the ψ-chain. The program in this case would go on by knitting the ψ-chain within the sub-program **C**.

B- Suppose the ϕ-chain is second class and is already terminated. In sub-program **B** we proceed to knitting the ψ-chain afterward. Suppose $\{\psi_i, \phi_1\}$ and $\{\psi_i, \psi_1\}$ both vanish. The ψ-chain terminates (and is first class) if $\{\psi_i, H_c\} \approx 0$ and it would be continued otherwise. Using lemma 3, one can show that conditions $\eta' \approx 0$ and $\gamma' \approx 0$ necessarily give $\gamma \approx 0$ and the ϕ-chain should have been terminated according to $\eta \not\approx 0$. Therefore, in this case, the ψ-chain is first class and the ϕ-chain is self-conjugate. If $\{\psi_j, H_c\} \not\approx 0$ then we should continue knitting the ψ-chain, but we should go first through step **F** which its necessity would be explained below. If γ' and/or η' does not vanish then the ψ-chain terminates and is second class. Different cases may happen depending on the non-vanishing set among η, γ, η' and η'. These cases will be considered in sub-programs **D** and **E**.

C- Suppose the ϕ-chain is first class and is terminated. Then the ψ-chain would be knitted independently. This is done in sub-program **C**, which is almost the same as what we did for one chain system. The ψ-chain may be first or second class and will be terminated in the usual manner in each case.

D- If $\gamma' \not\approx 0$ and $\gamma \not\approx 0$ (right branch in the flowchart **D**) then one can redefine ϕ_N and ψ_M to insure that $\eta \approx 0$ and $\eta' \approx 0$. So we have two second class chains in such a way that the terminating element of each chain does not commute with the top element of the other chain. We call this second class system a *cross-conjugate* (CC) two-chain system. In lemma 4 we show that the chains have the same length and their elements are pairwise conjugate. If $\gamma' \not\approx 0$ but $\gamma \approx 0$ (necessarily $\eta \not\approx 0$) then using the redefinition shown in the flowchart we make γ' to vanish and then come back to stage **H** of the sub-program **B**.

E- Suppose $\gamma' \approx 0$ and $\eta' \not\approx 0$. If $\gamma \approx 0$, then necessarily $\eta \not\approx 0$ and we have two self-conjugate second class chains. It will be shown in lemma 4 that by suitably redefining H_c the chains would commute (i.e. $\{\phi_i, \psi_j\} \approx 0$ for each ϕ_i and ψ_j). If $\gamma \not\approx 0$ one can make it to vanish by redefining ϕ_i as indicated in the flowchart. This alteration may change η; if it still does not vanish, the situation is like above and we have two self-conjugate chains. If η vanishes as a consequence of redefining ϕ_i, one should come back to the stage **F** of the sub-program **A** and continue knitting the ϕ-chain.

F- An essential point is that during knitting the ψ-chain, it may happen that the previously non-vanishing functions η or γ do vanish on the surface of new constraints. If so, one should come back to ϕ-chain and continue knitting it. Therefore, when a new constraint emerges in ψ-chain we should be sure that η and γ are still non vanishing. If at least one of them is remained nonzero then we are allowed to go to the beginning point of sub-program **B** and go on.

As is seen, after running the program we may have one of the following possibilities for the two chain system:

i) two first class chains,

ii) one first class and one self-conjugate chain,

iii) two self-conjugate chains,

iv) a system of two cross-conjugate chains.

In cases i-iii the constraints of each chain commute with the constraints of the other chain, but in case iv each constraint in one chain is conjugate to its partner in the other chain.

Now let us give some details that we encountered in the program as three lemmas.

lemma 2: If the ϕ-chain is FCC (relation (18)), then $\{\phi_i, \psi_j\} \approx 0$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$.

proof: Since the ϕ-chain is first class, it is obvious that $\{\phi_i, \psi_1\} \approx 0$ for $i = 1, \ldots, N$. Assuming $\{\phi_i, \psi_j\} \approx 0$ for a given ψ_j and all ϕ_i, we prove that $\{\phi_i, \psi_{j+1}\} \approx 0$. To do this, note that

$$\{\phi_i, \psi_{j+1}\} = \{\{\phi_i, \psi_j\}, H_c\} - \{\phi_{i+1}, \psi_j\} \tag{23}$$

where we have used the Jacobi identity and recursion relations (20) and (21). The second term on the RHS of (23) vanishes weakly according to our assumption. (In case $i = N$, one should consider a combination of ϕ_i's instead of ϕ_{N+1}.) In the first term on RHS of (23) $\{\phi_i, \psi_j\}$ is a combination of constraints and has weakly vanishing Poisson bracket with H_c. The only exemption is when $j = M$ and the ψ-chain is second class. In this case one can redefine H_c as follows:

$$H_c \rightarrow H_c - \frac{\{\psi_M, H_c\}}{\{\psi_M, \psi_1\}} \psi_1 \tag{24}$$

This redefinition ensures us that $\{\psi_M, H_c\} \approx 0$ and the lemma holds always.

lemma 3: If $\gamma \approx 0$,$\gamma' \approx 0$,$\eta \not\approx 0$ and $\eta' \not\approx 0$, then the chains can be made to commute.

proof: Redefining H_c as

$$H_c \to H_c - \frac{\{\phi_N, H_c\}}{\{\phi_N, \phi_1\}}\phi_1 - \frac{\{\psi_M, H_c\}}{\{\psi_M, \psi_1\}}\psi_1 \tag{25}$$

we have $\{\phi_N, H_c\} \approx 0$ and $\{\psi_M, H_c\} \approx 0$. Then following the same steps as we did in lemma 2, this lemma will also be proved.

lemma 4: If $\gamma \not\approx 0$, $\gamma' \not\approx 0$, $\eta \approx 0$ and $\eta' \approx 0$, then the chains have the same length and their constraints are pairwise conjugate to each other.
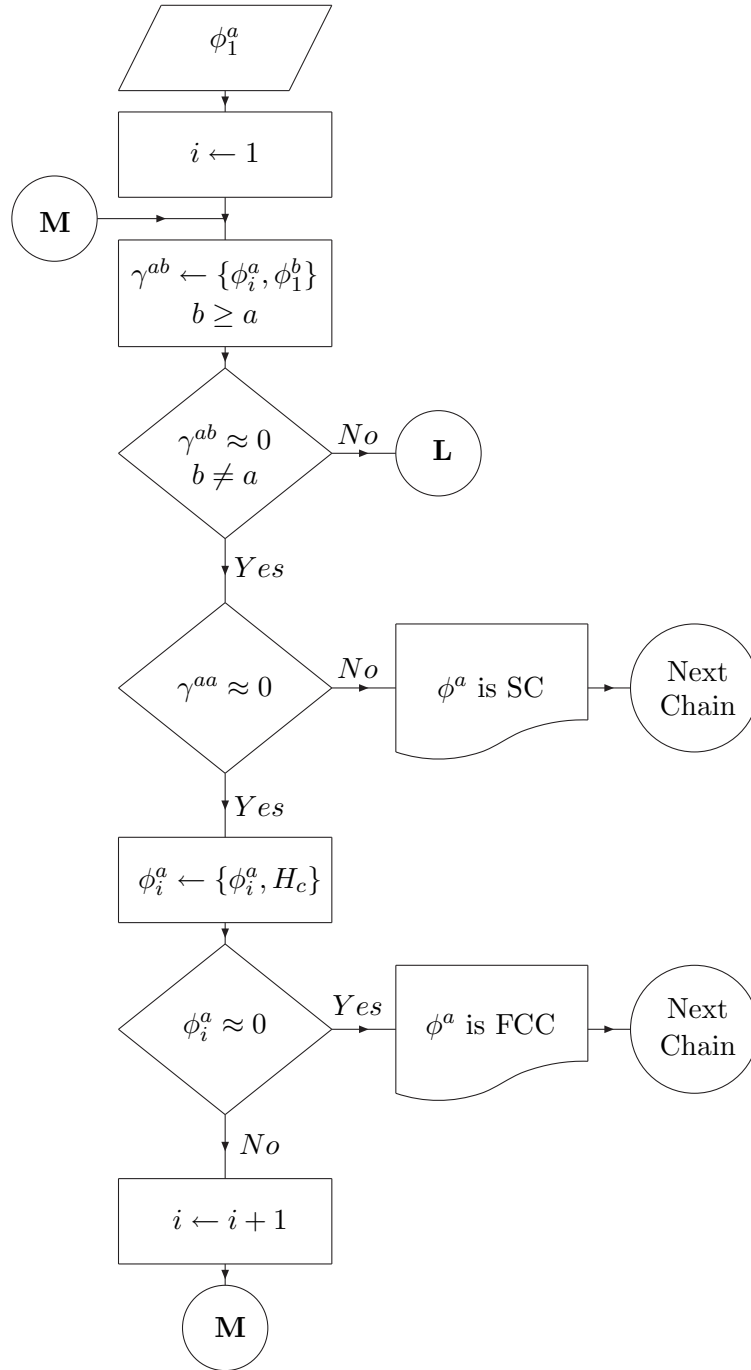
proof: Using Jacobi identity and recursion relations (20) and (21) it is easy to show that

$$\{\phi_N, \psi_1\} \approx -\{\phi_{N-1}, \psi_2\} \approx \ldots \approx (-1)^N\{\phi_1, \psi_N\} \tag{26}$$

This relation shows that the ψ-chain really terminates after N steps (i.e. $M = N$) and the conjugate pairs are as follows: $(\phi_N, \psi_1), (\phi_{N-1}, \psi_2), \ldots, (\phi_1, \psi_N)$.

4   Multi-chain System

Suppose we are given m primary constraints ϕ_1^a, $a = 1, \ldots, m$. The process of constructing the constraint chains is roughly a generalization of what we did for two-chain systems. However, one should first arrange the primary constraints in such a way that the set of primary second class constraints come first as cross-conjugate pairs. Suppose we have knitted constraint chains up to some particular chain which begins with ϕ_1^a. We want to show how the typical chain ϕ^a is produced. We assume that ϕ_1^a commute with all previous constraints; otherwise the chain ϕ^a should be conjugate to one of the previous chains. The following flowchart shows the main process of constructing the chain ϕ^a.

ϕ_1^a

$i \leftarrow 1$

M

$\gamma^{ab} \leftarrow \{\phi_i^a, \phi_1^b\}$
$b \geq a$

$\gamma^{ab} \approx 0$
$b \neq a$

No → **L**

Yes

$\gamma^{aa} \approx 0$

No → ϕ^a is SC → Next Chain

Yes

$\phi_i^a \leftarrow \{\phi_i^a, H_c\}$

$\phi_i^a \approx 0$

Yes → ϕ^a is FCC → Next Chain

No

$i \leftarrow i + 1$

M

In this process (except at step **M** which will be explained) we have nothing to do with the previous chains. This means that to treating with some constraint ϕ_1^a, it is sufficient to consider its Poisson brackets with ϕ_1^b's for $b \geq a$. There are some small differences with two-chain systems at steps **L** and **M** which will be explained below.

L- Suppose γ^{ab} defined as $\{\phi_1^a, \phi_1^b\}$ does not vanish for some $b \neq a$. One can move ϕ_1^b to the position next after ϕ_1^a and knit the corresponding chain. The system of chains ϕ^a and ϕ^b is a two-chain system, which should be analyzed in the same way as mentioned in the previous section. If more than one primary constraint do not commute with ϕ_1^a, then one can choose one of them and redefine the other ones in a way similar to appendix A to make them commute with ϕ_1^a.

M- Similar to the step **F** of the two-chain system, when a new constraint ϕ_i^a emerges, one should check whether the previously assumed non vanishing functions like η's, γ's, etc, (which appear in the end points of second class chains) are still non vanishing or not. For example, if we have had a self-conjugate chain ϕ^c, i.e. $\gamma^{cc} = \{\phi_{N_c}^c, \phi_1^c\}$ have been non vanishing and it vanishes after the emergence of a new constraint one should come back to the chain ϕ^c and continue to knit it.

After all, the set of constraints would be classified as a number of first class chains, self-conjugate second class chains and some pairs of cross-conjugate second class chains. As it is shown through the lemmas, it is easy to make the chains to commute with each other (only chains in a pair of cross-conjugate chains do not commute). Using the methods given in the appendices A and B, it is also possible, to redefine the second class chains such that their Poisson brackets obey the symplectic algebra.

5   Examples

it i) Consider the Lagrangian

$$L = \dot{x}\dot{z} + \frac{1}{2}\dot{\alpha}^2 + xy + \alpha\beta. \tag{27}$$

The primary constraints are

$$\phi_1 = p_y \quad , \quad \psi_1 = p_\beta, \tag{28}$$

and the canonical Hamiltonian can be written as

$$H_c = p_x p_z + \frac{1}{2} p_\alpha^2 - xy - \alpha\beta \tag{29}$$

By knitting the chains, we have two constraint chains as follows

$$
\begin{array}{ll}
\phi_1 = p_y & \psi_1 = p_\beta \\
\phi_2 = x & \psi_2 = \alpha \\
\phi_3 = p_z & \psi_3 = p_\alpha \\
& \psi_4 = \beta
\end{array} \tag{30}
$$

Clearly, the first chain is FC and the second one is self-conjugate.

ii) As an example of cross-conjugate chains, consider the system given by

$$L = \dot{x}\dot{z} + \dot{\alpha}\dot{\beta} + xy + \alpha\gamma + z\gamma. \tag{31}$$

Primary constraints are

$$\phi_1 = p_y \quad , \quad \phi_2 = p_\gamma \tag{32}$$

and the canonical Hamiltonian is

$$H_c = p_x p_z + p_\alpha p_\beta - xy - \alpha\gamma - z\gamma. \tag{33}$$

It is obvious that the chains

$$
\begin{array}{ll}
\phi_1 = p_y & \psi_1 = p_\gamma \\
\phi_2 = x & \psi_2 = z + \alpha \\
\phi_3 = p_z & \psi_3 = p_x + p_\beta \\
\phi_4 = \gamma & \psi_4 = y
\end{array} \tag{34}
$$

are cross-conjugate.

iii) As an example of multi chain system, consider the QED Lagrangian:

$$L = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\gamma^\mu (i\partial_\mu - eA_\mu)\psi - m\overline{\psi}\psi \right]. \tag{35}$$

Primary constraints and the canonical Hamiltonian are as follows:

$$\phi_1^1 = \Pi^0, \quad \phi_1^2 = \overline{\Pi}, \quad \phi_1^3 = \Pi - i\overline{\psi}\gamma^0, \tag{36}$$

$$H_c = \int d^3x \left[\frac{1}{2}\Pi_i^2 + A_0\partial_i\Pi_i + \frac{1}{4}F_{ij}F_{ij} - \overline{\psi}i\gamma^i\partial_i\psi - e\overline{\psi}\gamma^\mu A_\mu\psi + m\overline{\psi}\psi \right], \tag{37}$$

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where Π^μ, Π and $\overline{\Pi}$ are momenta conjugate to A^μ, ψ and $\overline{\psi}$ respectively.

Since $\{\phi_1^2, \phi_1^3\} = i\gamma^0$, these constraints form a cross conjugate pair. Consequently, according to lemma 3, we should redefine the canonical Hamiltonian H_c (relation 25),

$$H_c' = H_c - \phi_1^3 \frac{1}{\{\phi_1^2, \phi_1^3\}} \{\phi_1^2, H_c\} - \{\phi_1^3, H_c\} \frac{1}{\{\phi_1^3, \phi_1^2\}} \phi_1^2.$$

As mentioned before, this is necessary in order that the remaining chain commute with ϕ_1^2 and ϕ_1^3. Consequently, the ϕ^1-chain arises as

$$\begin{aligned} \phi_1^1 &= \Pi^0 \\ \phi_2^1 &= -\partial_i \Pi_i - ie(\Pi\psi - \overline{\psi}\,\overline{\Pi}). \end{aligned} \tag{38}$$

The interesting point is that without redefining H_c the ϕ^1-chain would terminate at $\phi_2'^1 = -\partial_i \Pi_i + e\overline{\psi}\gamma^0\psi$. This constraint does not commute with ϕ_1^2 and ϕ_1^3. In the literature [6] the problem is solved by searching for a first class combination of ϕ_1^2, ϕ_1^3 and $\phi_2'^1$. As we see, this difficulty is removed automatically in our method.

6   Conclusion

Given the primary constraints, the main task, in a constrained system, is to find the set of secondary constraints in such a way that the consistency of all constraints is achieved on the constraint surface, i.e. all the constraints commute weakly with H_c. The well known level by level method, explained in section 1, has some characteristics as follows:

1) At a given level, one should separate first and second class constraints from each other. This goal can be achieved in simple examples possessing a few number of constraints. However, for general cases, no method is introduced.

2) There is not a simple algebraic relation between the constraints of two adjacent levels, since finding the independent functions among $\{\phi_n^{a_1}, H_c\}$ (see the introduction) needs algebraic manipulations, which in the general case may combine these functions with each other.

3) It may happen that the rank of matrix $\{\phi_n^a, \phi_1^b\}$ reduces at subsequent levels. In other words, some of the previously assumed second class constraints may appear to be first class when the constraint surface extends due to emergence of new constraints. This important point, though considered in the literature [9, 10], but most of the time is bypassed by the

assumption that "the rank of the matrix of Poisson brackets of the constraints is constant during consistency process."

4) In general, the constraint algebra is not so simple. In other words, it is difficult to find simple algebraic relations between Poisson brackets of constraints. In fact, due to this difficulty, most studies on constrained systems have been done for special cases such as pure first class or pure second class systems.

We think that in chain by chain method, most of the above difficulties can be removed or reduced. In this method, one encounters some systematic operations (summarized as flowcharts) which can be followed easily. The second class constraints, in this method, are definitely separated from first class ones in a natural way, which makes their algebra very simple. In this way, most of the results, achieved through literature for purely first class systems or purely second class systems, can be easily generalized for arbitrary systems possessing both first and second class constraints.

As mentioned in introduction, the chain structure and irreducibility are the main results of our method. One important application is in constructing the generating function of gauge transformations, which is a well known problem in the context of gauge theories [12]. For example, the authors of [13] have tried to prove the existence of the gauge generator, but they encountered some difficulties since the constraints in their chain structure are not necessarily irreducible. In some cases, such as for Abelian constraints, the chain structure makes it easy to solve the conditions that the gauge generator should satisfy. However, in our opinion, the chain by chain method is able to provide a new framework for discussing the problem of constructing the gauge generator for the general cases.

As another important application, the chain by chain method can be better applied in the gauge fixing process [14]. In that paper, we have shown that one should introduce only gauge fixing constraints which are conjugate to the last elements of the chains. Then, their consistency would give us the other necessary gauge fixing constraints. As is apparent, such a simple method for fixing the gauges can not be found in the context of level by level method.

Acknowledgements

Appendix A

Here we prove that for a 1-chain system the constraints can be put in a way so that their algebra has the form given by (14,15). Let us first rename the constraints in the second half of the chain such that

$$\phi_i^* = (-1)^i \phi_{N-i+1} \qquad i = 1, \ldots, K. \tag{39}$$

In this way lemma 1 reads

$$\{\phi_i, \phi_j\} \approx 0 \qquad i, j = 1, \ldots, K \tag{40}$$

$$\{\phi_i, \phi_j^*\} \approx 0 \qquad i = 1, \quad \ldots, K, \ j > i \tag{41}$$

$$\{\phi_i, \phi_i^*\} \approx \eta. \tag{42}$$

Now we can redefine the set $(\phi_1, \ldots, \phi_K; \phi_1^*, \ldots, \phi_K^*)$ such that

$$\tilde{\phi}_1 = \phi_1,$$
$$\tilde{\phi}_1^* = \phi_1^*, \tag{43}$$

$$\tilde{\phi}_i = \phi_i - \sum_{k=1}^{i-1} \frac{\{\phi_i, \tilde{\phi}_k^*\}}{\{\tilde{\phi}_k, \tilde{\phi}_k^*\}} \tilde{\phi}_k, \tag{44}$$

$$\tilde{\phi}_i^* = \phi_i^* - \sum_{k=1}^{i-1} \frac{\{\phi_i^*, \tilde{\phi}_k^*\}}{\{\tilde{\phi}_k, \tilde{\phi}_k^*\}} \tilde{\phi}_k. \tag{45}$$

Using (40) it is easy to show that

$$\{\tilde{\phi}_i, \tilde{\phi}_j\} \approx 0. \tag{46}$$

We can also show that

$$\{\tilde{\phi}_i, \tilde{\phi}_i^*\} \approx \delta_{ij} \eta. \tag{47}$$

To prove this assertion, consider a definite j. For $i < j$, using (45) and (46) it is obvious that $\{\tilde{\phi}_i, \tilde{\phi}_j^*\} \approx 0$. For $i = j$ using (44-46) and (42) one can write

$$\{\tilde{\phi}_i, \tilde{\phi}_i^*\} \approx \{\phi_i, \phi_i^*\}$$
$$\approx \eta \tag{48}$$

For $i > j$ we prove (47) inductively. First one can see that (47) is true for $i = 2$ and $j = 1$, i.e.

$$\{\tilde{\phi}_2, \tilde{\phi}_1^*\} = \left\{ \phi_2 - \frac{\{\phi_2, \phi_1\}}{\{\phi_1, \phi_N\}}\phi_1 \ , \ \phi_N \right\} \approx 0, \tag{49}$$

where we have used (43),(44) and (39). Suppose that (47) holds at all steps up to a definite step i. This means that

$$\{\tilde{\phi}_k, \tilde{\phi}_j^*\} \approx 0 \qquad k \neq j, \ k = 2, \ldots, i. \tag{50}$$

Then we show that (50) is also true for $k = i + 1$. For this reason, consider

$$\{\tilde{\phi}_{i+1}, \tilde{\phi}_j^*\} = \left\{ \phi_{i+1} - \sum_{k=1}^{i} \frac{\{\phi_{i+1}, \tilde{\phi}_k^*\}}{\{\tilde{\phi}_k, \tilde{\phi}_k^*\}}\tilde{\phi}_k \ , \ \tilde{\phi}_j^* \right\}. \tag{51}$$

Using (50) only the term $k = j$ remains in the sum over k, which would be cancelled by construction, to give

$$\{\tilde{\phi}_{i+1}, \tilde{\phi}_j^*\} \approx 0. \tag{52}$$

Finally, we prove that

$$\{\tilde{\phi}_i^*, \tilde{\phi}_j^*\} \approx 0. \tag{53}$$

Suppose $i > j$. Using (44) for $\tilde{\phi}_i^*$ and then (48), the proof is straightforward. Putting (46), (47) and (53) altogether, the desired algebra (14) of the text would be obtained by defining

$$\begin{aligned} \Omega_i &= \eta^{-1}\tilde{\phi}_i \\ \Omega_{K+i} &= \tilde{\phi}_i^* \end{aligned} \qquad i = 1, \ldots, K \tag{54}$$

Appendix B

In this appendix, we generalize the results of appendix A. Consider the two cross-conjugate chains

$$\begin{matrix} \phi_1 & \psi_1 \\ \vdots & \vdots \\ \phi_N & \psi_N \end{matrix} \tag{55}$$

with algebra

$$\{\phi_N, \psi_1\} \approx (-1)^N \{\psi_N, \phi_1\} \approx \eta \tag{56}$$

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If N is even one can replace ϕ_1 and ψ_1 with $\xi_1 = \phi_1 + \psi_1$ and $\zeta_1 = \phi_1 - \psi_1$. This would result to the chains

$$
\begin{array}{cc}
\xi_1 = \phi_1 + \psi_1 & \zeta_1 = \phi_1 - \psi_1 \\
\vdots & \vdots \\
\xi_1 = \phi_N + \psi_N & \zeta_N = \phi_N - \psi_N
\end{array}
\tag{57}
$$

In this way the algebra of the constraints is changed to

$$
\begin{aligned}
\{\xi_N, \zeta_1\} &\approx \{\zeta_N, \xi_1\} \approx 0 \\
\{\xi_N, \xi_1\} &\approx \{\zeta_N, \zeta_1\} \approx 2\eta.
\end{aligned}
\tag{58}
$$

As is observed, we have replaced a pair of cross-conjugate chains with two self-conjugate ones. The remainder of the procedure is as in the case of one chain system. That is, following the steps given in appendix A, one can reach to a symplectic algebra among the constraints. As an example, the reader can test the Lagrangian (31) with chains given in (34).

Now suppose N is odd. If $\{\phi_N, H_c\} \approx 0$, from lemma 3, the elements of the ϕ-chain commute with each other. Noticing (26) one can see that by defining ϕ_i^* as

$$
\phi_i^* = (-1)^i \psi_{N-i+1} \quad i = 1, \ldots, N
\tag{59}
$$

the same algebra of (40)-(42) will be reproduced. Therefore, one can follow the procedure of appendix A to reach the desired goal. If $\{\phi_N, H_c\} \not\approx 0$, but instead $\{\psi_N, H_c\} \approx 0$, the same thing can be done, this time with redefining ϕ_i's as

$$
\psi_i^* = (-1)^i \phi_{N-i+1}.
\tag{60}
$$

The only considerable case occurs when

$$
\begin{aligned}
\{\phi_N, H_c\} &= \gamma \not\approx 0 \\
\{\psi_N, H_c\} &= \chi \not\approx 0
\end{aligned}
\tag{61}
$$

This time we consider

$$
\xi_1 = \chi \phi_1 - \gamma \psi_1
\tag{62}
$$

as the primary constraint of the first chain. After $N-1$ levels of consistency, one would obtain

$$
\xi_N = \chi \phi_N - \gamma \psi_N
\tag{63}
$$

such that

$$
\{\xi_N, H_c\} \approx 0
\tag{64}
$$

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as can be seen directly from (61). Moreover, using (56) one can see

$$\{\xi_N, \xi_1\} \approx \chi\gamma(1 + (-1)^N)\eta \tag{65}$$

which vanishes for N odd. Again the elements of the ξ-chain commute with each other, so we continue like the previous case.

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