

Finite Order BFFT Method

M. Monemzadeh^{a*}

A. Shirzad^{a,b†}

^a *Department of Physics, Isfahan University of Technology (IUT)
Isfahan, Iran,*

^b *Institute for Studies in Theoretical Physics and Mathematics (IPM)
P. O. Box: 19395-5531, Tehran, Iran.*

Abstract

We have proposed a method in the context of BFFT approach that leads to truncation of the infinite series regarded to constraints in the extended phase space, as well as other physical quantities (such as Hamiltonian). This has been done for cases where the matrix of Poisson brackets among the constraints is symplectic or constant. The method is applied to Proca model, single self dual chiral bosons and chiral Schwinger models as examples.

1 Introduction

The Dirac procedure is well-known for canonical quantization of the first class constrained systems [1]. The corresponding analysis in the path integral approach was also initiated by Faddeev for gauge theories [2]. To quantize a second class constrained system in Dirac approach, it is necessary to replace Poisson brackets by Dirac brackets. Converting Dirac brackets to quantum commutators sometimes implies factor ordering problem and quantization of these models is not formal. Batalin and his collaborators proposed the conversion of the second class constraints into first class ones by defining a set of new auxiliary variables [3, 4]. In this method (the BFFT method) one can find correction terms for constraints and Hamiltonian in an iterative process, the first correction is linear in the new variables, the second is quadratic and so on. In this way one obtains a gauge

*e-mail: monemzadeh@sepahan.iut.ac.ir

†shirzad@ipm.ir

theory and then applies the well-known mechanisms for their quantization [2, 5, 6, 7]. It is important to notice that this idea is a logical following of the original notion of Stückelberg who converted second class theories to first class ones by extending the configuration space with some scalar fields (Stückelberg scalars) [8].

In this paper we show that there exist some arbitrary parameters that if suitably chosen then the series of the correction terms of constraints and Hamiltonian do terminate. We call this approach "the finite order BFFT method". In section 2 we briefly review the essence of the BFFT method. Without losing the generality we assume a system with second class constraints only. In section 3 we show that in principle it is possible to chose the arbitrary parameters in such a way that the correction terms terminate, provided that the matrix of Poisson brackets of constraints is either symplectic or constant. We apply our process to the Proca model, single self dual chiral bosons and chiral Schwinger Model in sections 4,5 and 6 respectively. Section 7 is devoted to conclusions.

2 Brief Review of the BFFT Formalism

Consider a second class constrained system described by Hamiltonian H_0 in phase space with coordinates (q^i, p_i) where $i = 1, 2, \dots, K$. Assume the system is under the influence of a set of second class constraints, Θ_α $\alpha = 1, \dots, m$, satisfying the algebra

$$\Delta_{\alpha\beta} = \{\Theta_\alpha, \Theta_\beta\} \quad (1)$$

where $\{, \}$ means Poisson bracket and $\Delta_{\alpha\beta}$ is an invertible matrix. For converting a second class system into a true gauge system one can enlarge the phase space by introducing auxiliary variables, one for each constraint. We denote the variables by η^α with the following algebra:

$$\{\eta^\alpha, \eta^\beta\} = \omega^{\alpha\beta} \quad (2)$$

where $\omega^{\alpha\beta}$ is an antisymmetric matrix which we *assume* it to be constant. The first class constraints in the extended phase space $(q, p) \oplus (\eta)$ are defined by

$$\tau_\alpha = \tau_\alpha(q, p, \eta) \quad \alpha = 1, 2, \dots, m \quad (3)$$

with the boundary conditions

$$\tau_\alpha(q, p, 0) = \Theta_\alpha(q, p). \quad (4)$$

In the abelian BFFT embedding method one demands that these extended constraints are strongly involutive:

$$\{\tau_\alpha, \tau_\beta\} = 0. \quad (5)$$

The solution of the above equation can be obtained by considering τ_α as:

$$\tau_\alpha = \sum_{n=0}^{\infty} \tau_\alpha^{(n)} \quad (6)$$

where $\tau_\alpha^{(n)}$ is of order n with respect to η^α 's. According to the boundary condition (4) we have

$$\tau_\alpha^{(0)} = \Theta_\alpha. \quad (7)$$

Substituting Eq. (6) into Eq. (5) leads to a set of recursive relations. Vanishing the term independent of η gives:

$$\{\tau_\alpha^{(0)}, \tau_\beta^{(0)}\} + \{\tau_\alpha^{(1)}, \tau_\beta^{(1)}\}_{(\eta)} = 0; \quad (8)$$

and vanishing the terms of order n with respect to η^α 's for $n \geq 1$ gives

$$\{\tau_{[\alpha}^{(1)}, \tau_{\beta]}^{(n+1)}\}_{(\eta)} + B_{\alpha\beta}^{(n)} = 0 \quad n \geq 1 \quad (9)$$

where

$$B_{\alpha\beta}^{(1)} \equiv \{\tau_{[\alpha}^{(0)}, \tau_{\beta]}^{(1)}\} \quad (10)$$

and

$$B_{\alpha\beta}^{(n)} \equiv \frac{1}{2} B_{[\alpha\beta]} \equiv \sum_{m=0}^n \{\tau_\alpha^{(n-m)}, \tau_\beta^{(m)}\} + \sum_{m=0}^{n-2} \{\tau_\alpha^{(n-m)}, \tau_\beta^{(m+2)}\}_{(\eta)} \quad n \geq 2. \quad (11)$$

The suffix η in the above equations means that the Poisson brackets must be evaluated with respect to η variables only, otherwise they are calculated in the basis (q, p) . The above equations are used iteratively to obtain the correction terms $\tau^{(n)}$. Since $\tau^{(1)}$ is linear with respect to η we may write

$$\tau_\alpha^{(1)} = \chi_{\alpha\beta}(q, p)\eta^\beta. \quad (12)$$

Substituting this expression into Eq.(8) and using Eqs.(1) and (2) we obtain:

$$\Delta_{\alpha\beta} + \chi_{\alpha\gamma}\omega^{\gamma\lambda}\chi_{\beta\lambda} = 0. \quad (13)$$

This equation contains two unknown elements; $\chi_{\alpha\beta}$ and $\omega^{\alpha\beta}$. One should at first assume a suitable anti-symmetric matrix for $\omega^{\alpha\beta}$ and then solve Eq. (13) to determine the coefficients $\chi_{\alpha\beta}$. Since $\Delta_{\alpha\beta}$ and $\omega^{\alpha\beta}$ are anti-symmetric matrices, there are totally $\frac{m(m-1)}{2}$ independent equations for $\chi_{\alpha\beta}$, while the number of $\chi_{\alpha\beta}$'s are m^2 . Therefore there exist an infinite number of solutions for $\chi_{\alpha\beta}$ and we are allowed to chose any solution we wish. Using this possibility, $\chi_{\alpha\beta}$'s can be chosen such that the process of determining the correction terms $\tau^{(n)}$ terminate at this stage, i.e. $\tau^{(2)}$ vanishes. We will come to this point in the next section. It can be seen that the general solution of Eq. (9) is given by [9]

$$\tau_\alpha^{(n+1)} = -\frac{1}{n+2}\eta^\mu\omega_{\mu\nu}\chi^{\nu\rho}B_{\rho\alpha}^{(n)}; \quad n \geq 1 \quad (14)$$

where $\omega_{\alpha\beta}$ and $\chi^{\alpha\beta}$ are inverse to $\omega^{\alpha\beta}$ and $\chi_{\alpha\beta}$ respectively.

To construct the corresponding Hamiltonian $\tilde{H}(q, p, \eta)$ in the extended phase space we demand

$$\tilde{H}(q, p, 0) = H(q, p) \quad (15)$$

and

$$\{\tau_\alpha, \tilde{H}\} = 0. \quad (16)$$

Similar to τ_α , suppose

$$\tilde{H} = \sum_{n=0}^{\infty} \tilde{H}^{(n)} \quad (17)$$

where $\tilde{H}^{(n)}$ is of order n with respect to η^α 's and

$$\tilde{H}^{(0)} = H(q, p). \quad (18)$$

Substituting from Eqs. (6) and (17) in Eq. (16) gives:

$$\{\tau_\alpha^{(1)}, \tilde{H}^{(n+1)}\}_{(\eta)} + G_\alpha^{(n)} = 0; \quad n \geq 0 \quad (19)$$

where $G_\alpha^{(n)}$ as the generators of the $\tilde{H}^{(n+1)}$ are defined as follow

$$G_\alpha^{(0)} \equiv \{\tau_\alpha^{(0)}, \tilde{H}^{(0)}\} \quad (20)$$

$$G_\alpha^{(1)} \equiv \{\tau_\alpha^{(1)}, \tilde{H}^{(0)}\} + \{\tau_\alpha^{(0)}, \tilde{H}^{(1)}\} + \{\tau_\alpha^{(2)}, \tilde{H}^{(1)}\}_{(\eta)} \quad (21)$$

$$G_\alpha^{(n)} \equiv \sum_{m=0}^n \{\tau_\alpha^{(n-m)}, \tilde{H}^{(m)}\} + \sum_{m=0}^{n-2} \{\tau_\alpha^{(n-m)}, \tilde{H}^{(m+2)}\}_{(\eta)} + \{\tau_\alpha^{(n+1)}, \tilde{H}^{(1)}\}_{(\eta)}; \quad n \geq 2. \quad (22)$$

It can be shown that the general expression for $\tilde{H}^{(n)}$ is

$$\tilde{H}^{(n+1)} = -\frac{1}{n+1} \eta^\alpha \omega_{\alpha\beta} \chi^{\beta\nu} G_\nu^{(n)}. \quad (23)$$

Similarly for every function $F(q, p)$ in the phase space one can write

$$\tilde{F}(q, p, \eta) = \sum_{n=0}^{\infty} \tilde{F}^{(n)}, \quad (24)$$

where $\tilde{F}^{(n)}$ is of order n with respect to η^α 's and

$$\tilde{F}^{(n+1)} = -\frac{1}{n+1} \eta^\alpha \omega_{\alpha\beta} \chi^{\beta\nu} \rho_\nu^{(n)}. \quad (25)$$

In this relation $\rho_\nu^{(n)}$ can be derived similar to $G_\nu^{(n)}$ in Eqs. (20-22) by replacing H with F .

This completes the BFFT construction of the first class system which is strongly involutive. As can be seen the correction terms of $\tau_\alpha^{(n)}$ and $\tilde{H}^{(n)}$ are derived iteratively from Eqs.(14) and (23). Generally, there is no guarantee that the series terminate at some definite order. However, the series will terminate if $B_{\alpha\beta}^{(N)}$ and $G_\alpha^{(N)}$ vanish for a certain order $n = N$.

3 Finite Order Method

In this section we want to solve the iterative equations for $\tau_\alpha^{(n)}$ and $\tilde{H}^{(n)}$ in such a way that the corresponding series terminate as soon as possible. We remember that $\omega^{\alpha\beta}$ can be chosen arbitrarily. On the other hand Eq. (13) for $\chi_{\alpha\beta}$'s is not so much restrictive. We use these possibilities to find a systematic method to truncate infinite series encountered in BFFT method. However, the problem seems difficult for a general second class system. In the following we solve it for two special cases, i.e. where the matrix $\Delta_{\alpha\beta}$ given in (1) is symplectic or constant.

A- Suppose $\Delta_{ij} = J_{ij}$, where J is the symplectic matrix:

$$J = \left(\begin{array}{c|c} \mathbf{0} & -\mathbf{1} \\ \hline +\mathbf{1} & \mathbf{0} \end{array} \right).$$

In principle it has been shown that one can usually redefine the second class constraints as pairs of coordinates and momenta with the symplectic algebra [10, 11]. The algebra of the new variables η^α and unknown coefficients $\chi_{\alpha\beta}$ can be chosen as

$$\begin{aligned} \omega^{\alpha\beta} &= \{\eta^\alpha, \eta^\beta\} = \tilde{J}_{\alpha\beta} = -J_{\alpha\beta} \\ \chi_{\alpha\beta} &= J_{\alpha\beta} \end{aligned} \quad (26)$$

It is easy to check that ω and χ in Eq. (26) satisfy the basic equation Eq. (13) for $\Delta = J$. So the first correction term of the constraints is

$$\tau_\alpha^{(1)} = \chi_{\alpha\beta} \eta^\beta = J_{\alpha\beta} \eta^\beta. \quad (27)$$

Since $\tau_\alpha^{(1)}$ is only a function of η , it can be seen in a straightforward way that $B_{\alpha\beta}^{(n)}$ vanish for $n \geq 1$. As a result τ_α series terminate at this step. The new set of constraints are found to be

$$\tau_\alpha(q, p, \eta) = \tau_\alpha(q, p)^{(0)} + J_{\alpha\beta} \eta^\beta. \quad (28)$$

One can directly check that τ_α 's are strongly involutive. To complete our procedure we should also construct the extended Hamiltonian. Inserting (26) into (23), the correction terms of Hamiltonian are deduced as

$$\tilde{H}^{(n+1)} = -\frac{1}{n+1} \eta^\alpha G_\alpha^{(n)} \quad (29)$$

It is necessary to evaluate the $G_\alpha^{(n)}$ as the generators of \tilde{H} ; i.e. $\tilde{H}^{(n+1)} \sim G_\alpha^{(n)}$. For the the zeroth order we have

$$G_\mu^{(0)} = \{\tau_\mu^{(0)}, H_0\}. \quad (30)$$

The next correction term for \tilde{H} is

$$\tilde{H}^{(1)} = -\eta^\mu G_\mu^{(0)}. \quad (31)$$

This should be inserted into Eq. (21) to find

$$G_\alpha^{(1)} = -\eta^\mu C_{\alpha\mu} \quad (32)$$

where

$$C_{\alpha\mu} = \{\tau_\alpha^{(0)}, \{\tau_\mu^{(0)}, H_0\}\}. \quad (33)$$

Similarly $\tilde{H}^{(2)}$ can be derived from Eq. (23) as

$$\tilde{H}^{(2)} = \frac{1}{2}\eta^\mu\eta^\nu C_{\mu\nu}. \quad (34)$$

This process continue until $G_\alpha^{(n)}$ become a function of η 's only. If H_0 is at most quadratic with respect to phase space coordinates, it would be clear that $C_{\alpha\mu}$ in Eq. (33) is constant and $G_\alpha^{(2)} = 0$; and consequently $\tilde{H}^{(3)} = 0$. In this case one can finally write

$$\tilde{H} = H_0 - \eta^\mu C_\mu^{(0)} + \frac{1}{2}\eta^\mu\eta^\nu C_{\mu\nu}. \quad (35)$$

In a more general case, when H_0 is a function of order N with respect to coordinates (q, p) and the constraints are linear with respect to coordinates and momenta, the series of \tilde{H} will be finished at N th step; i.e.

$$\tilde{H} = H_0 + \tilde{H}^{(1)} + \dots + \tilde{H}^{(N)} \quad (36)$$

Eqs. (28) and (36) represent a finite order gauge theory in abelian BFFT approach. In this way we can convert every second class constraint system to a *rank zero gauge theory*, in which the structure functions $C_{\alpha\beta}^\gamma$ and V_α^β defined in

$$\begin{aligned} \{\tau_\alpha, \tau_\beta\} &= C_{\alpha\beta}^\gamma \tau_\gamma \\ \{\tau_\alpha, \tilde{H}\} &= V_\alpha^\beta \tau_\beta \end{aligned} \quad (37)$$

vanish in the extended phase space [7].

Assuming again that Δ is the symplectic matrix, one can also select $\omega = \Delta^T = -J$. Then the basic Eq. (13) implies that

$$J = \chi^T J \chi. \quad (38)$$

As stated in Eq.(26), $\chi = J$ satisfy the above equation. On the other hand, as is well-known [12], a canonical transformation from the set (q, p) to (Q, P) is represented by

$$J = M^T J M \quad (39)$$

where

$$M = \frac{\partial(q, p)}{\partial(Q, P)}. \quad (40)$$

Comparing Eq. (39) with Eq. (38) shows that any canonical transformation in phase space of (q, p) can introduce a solution to the basic equation (13). In this way a large class of solutions are obtained, among them those with constant elements for M give truncated series for constraints.

B- In most physical examples of second class systems the Δ -matrix in (1) emerge as a matrix with constant elements. In this case we can choose

$$\omega = \Delta^T = -\Delta. \quad (41)$$

So the basic Eq. (13) can be written as

$$\Delta - \chi^T \Delta \chi = 0. \quad (42)$$

It is easy to see that $\chi = 1$ satisfies the above equation. Then the new set of constraints are of the form

$$\tau_\alpha = \tau_\alpha^{(0)} + \eta^\alpha. \quad (43)$$

The correction terms of the Hamiltonian can be derived as¹

$$\tilde{H}^{(n+1)} = \frac{1}{n+1} \eta^\alpha (\Delta^{-1})_\alpha^\beta G_\beta^{(n)} \quad (44)$$

where $G_\alpha^{(n)}$ are defined in Eqs. (20-22). For a Hamiltonian which is a polynomial of order N with respect to the original phase space coordinates (q, p) , the generators $G_\alpha^{(N)}$ will be only a function of auxiliary variables. Therefore the \tilde{H} series will terminate at N th step and the constraints (43) and \tilde{H} with correction terms (44) represent a rank zero gauge theory.

The significance of the above method can be better seen in the context of the chain by chain method introduced recently in [11]. Suppose we have only one chain of second class constraints with the recursion formula:

$$\Theta_{n+1} = \{\Theta_n, H_0\}. \quad (45)$$

Suppose Δ is a matrix with constant elements and we choose our arbitrary parameters ω and χ in such a way that the new set of constraint are given by Eq.(43). It is clear from (45) and (20) that

$$\begin{aligned} G_\alpha^{(0)} &= \Theta_{\alpha+1} & \alpha &= 1, 2, \dots, m-1 \\ G_m^{(0)} &= \{\Theta_m, H_0\}. \end{aligned} \quad (46)$$

¹Notice that the indices α, β, \dots have not tensorial mining. i.e. there is no metric to raise up or lower down the indices. Therefore the reader should not be worried about up-down indices on matrix Δ^{-1} in Eq. (44), etc.

So the first correction term of \tilde{H} is

$$\tilde{H}^{(1)} = \sum_{\beta=1}^{m-1} \eta^\alpha (\Delta^{-1})_\alpha^\beta \Theta_{\beta+1} + \eta^\alpha (\Delta^{-1})_\alpha^m \{\Theta_m, H_0\}. \quad (47)$$

It can be seen that

$$\tilde{H}^{(2)} = \sum_{\beta=1}^{m-1} \frac{1}{2} \eta^\mu \eta^\alpha (\Delta^{-1})_\mu^\nu (\Delta^{-1})_\alpha^\beta \Delta_{\nu\beta+1} + \frac{1}{2} \eta^\mu \eta^\alpha (\Delta^{-1})_\mu^\nu (\Delta^{-1})_\alpha^m \{\Theta_\nu, \{\Theta_m, H_0\}\}. \quad (48)$$

As we know from Eq.(23); $\tilde{H}^{(n+1)} \sim G_\alpha^{(n)}$ and

$$G_\alpha^{(n)} \sim \{\Theta_\alpha, \{\Theta_{\alpha_1}, \{\Theta_{\alpha_2}, \dots \{\Theta_{\alpha_n}, H_0\}\}\dots\}\}. \quad (49)$$

If H_0 is a polynomial of finite order N with respect to the phase space coordinates, then Eq. (49) shows that its correction terms do terminate at most after N steps.

Now we apply the above procedures to some definite models.

4 The Proca Model

As the first example we consider the Proca model, whose dynamics is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} A^\mu A_\mu \quad (50)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (51)$$

It is well-known that the second term in Eq. (50) breaks the gauge symmetry of the usual Maxwell's theory given by the first term. The canonical momenta are defined as

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu}(x). \quad (52)$$

From Eqs. (51) and (52) there is only one primary constraint field

$$\Theta_1(x) \equiv \pi^0(x) \approx 0 \quad (53)$$

where the symbol \approx means weak equality. The canonical Hamiltonian is

$$H_c = \int \left[\frac{1}{2} \pi_i^2 + \frac{1}{4} F_{ij}^2 + \frac{1}{2} (A_i^2 - A_0^2) - A_0 \partial_i \pi^i \right] d\mathbf{x}. \quad (54)$$

The total Hamiltonian is defined as

$$H_T = H_c + \int d\mathbf{x} \lambda(x) \Theta_1(x) \quad (55)$$

where $\lambda(x)$ is the Lagrange multiplier field. Following the algorithm of Dirac, we find that the consistency in time of the primary constraint (i.e. $\dot{\Theta}_1 = \{\Theta_1, H_C\} = 0$) leads to the secondary constraint field

$$\Theta_2(x) \equiv \partial_i \pi^i + A_0 \approx 0. \quad (56)$$

The consistency condition of Θ_2 just determines the Lagrange multiplier $\lambda(x)$. The algebra of the second class constraints in Eqs. (53) and (56) satisfy the basic condition² $\Delta_{ij} = J_{ij}$. For simplicity in our calculation we apply the following canonical transformation

$$\begin{aligned} A &= \partial_i \pi^i + A_0 \\ \pi &= \pi^0 \\ A'_i &= A_i + \partial_i \pi^0 \\ \pi'^i &= \pi^i. \end{aligned} \quad (57)$$

The new set of constraints and Hamiltonian are found to be

$$\Theta'_1(x) \equiv A(x) = 0 \quad \Theta'_2(x) \equiv \pi(x) = 0 \quad (58)$$

$$H'_c = \frac{1}{2} \pi'^i \pi'_i + \frac{1}{4} \tilde{F}^{ij} \tilde{F}_{ij} + \frac{1}{2} \left[(\partial_i \pi'^i)^2 + (\partial_i \pi)^2 - A^2 \right] - A'_i \partial_i \pi \quad (59)$$

where

$$\tilde{F}^{ij} = \partial^i A'^j - \partial^j A'^i. \quad (60)$$

In order to convert the above gauge non-invariant theory to a first class one, we make use of two new auxiliary fields η^1 and η^2 . According to Eq. (26) we choose

$$\begin{aligned} \omega^{\alpha\beta} &= \{\eta^\alpha, \eta^\beta\} = -J^{\alpha\beta} \\ \chi_{\alpha\beta} &= J_{\alpha\beta}. \end{aligned} \quad (61)$$

The first class constraints are deduced from Eq. (28) as

$$\tau_1 \equiv A + \eta^2 \quad \tau_2 \equiv \pi - \eta^1. \quad (62)$$

The generators in the first correction term of H'_c are

$$\begin{aligned} G_1^{(0)} &= \partial_i A'_i(x) - \partial_i \partial_i \pi(x) \\ G_2^{(0)} &= A(x) \end{aligned} \quad (63)$$

²Since the constraints are space-time fields, a three dimensional Dirac δ -function should be understood in Poisson bracket of constraints. More precisely we have

$$\{\Theta_i(\mathbf{x}, t), \Theta_j(\mathbf{y}, t)\} = \delta(\mathbf{x} - \mathbf{y}) J_{ij} \quad i, j = 1, 2.$$

However, we omit the δ -functions when not needed.

and from Eq. (31) one finds that

$$\tilde{H}'^{(1)} = \eta^1(\partial_i\partial_i\pi - \partial_i A'_i) - \eta^2 A. \quad (64)$$

Explicit calculations from Eq. (34) yield the last correction term as

$$\tilde{H}'^{(2)} = -\frac{1}{2}(\eta^1\partial_i\partial_i\eta^1 + \eta^2\eta^2). \quad (65)$$

So the embedded Hamiltonian is

$$\tilde{H}'_C = H'_C + \tilde{H}'^{(1)} + \tilde{H}'^{(2)}. \quad (66)$$

One can easily check that Eqs. (62) and (66) represent an abelian gauge theory.

5 Gauge-Invariant Single Self Dual Chiral Bosons

The gauge non-invariant Srivastava model for single self dual Chiral bosons in $(1 + 1)$ dimensions is described by the Lagrangian density [13]:

$$\mathcal{L}^N = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 + \lambda(\dot{\phi} - \phi') \quad (67)$$

where $\dot{\phi} \equiv \partial_0\phi$ and $\phi' \equiv \partial_1\phi$.

In this section we use the Lorentz metric $g^{\mu\nu} = \text{diag}(+1, -1)$. The canonical momenta can be derived as:

$$\pi = \dot{\phi} + \lambda \quad P_\lambda = 0 \quad (68)$$

where π and P_λ are the momenta conjugate to the fields ϕ and λ respectively. There is one primary constraint ($\Theta_1 \equiv P_\lambda \approx 0$). The canonical Hamiltonian density corresponding to \mathcal{L}^N is

$$\mathcal{H}_C^N = \frac{1}{2}(\pi - \lambda)^2 + \frac{1}{2}\phi'^2 + \lambda\phi'. \quad (69)$$

Consistency condition of the primary constraint leads to a secondary constraint

$$\Theta_2 \equiv \pi - \phi' - \lambda \approx 0. \quad (70)$$

Since $[\Theta_1, \Theta_2] \neq 0$ the constraint chain finishes at this step. We have two second class constraints satisfying the symplectic algebra which represent a gauge non-invariant model. This model was considered in Stückelberg method with enlarging the Hilbert space of the theory and introducing a full quantum field θ , called Wess- Zumino field [14], to obtain the modified Lagrangian density as:

$$\mathcal{L}^I = \mathcal{L}^N + \mathcal{L}^{WZ}; \quad \mathcal{L}^{WZ} = -\frac{1}{2}(\dot{\theta} + \theta'^2) + \theta'(\phi' + \dot{\theta}) - \dot{\theta}(\phi' + \lambda) + \lambda\theta'. \quad (71)$$

In this section we concentrate on this model in BFFT method and introduce η^1 and η^2 as auxiliary fields with the algebra

$$\omega^{\alpha\beta} = \{\eta^\alpha, \eta^\beta\} = -J^{\alpha\beta}. \quad (72)$$

According to the procedure defined before and Eq.(28) the new abelian first class constraints are:

$$\begin{aligned} \tau_1 &\equiv P_\lambda + \eta^2 \\ \tau_2 &\equiv \pi - \phi' - \lambda - \eta^1. \end{aligned} \quad (73)$$

The embedded Hamiltonian density in the extended phase space with the mention to (30-34) are derived as

$$\tilde{\mathcal{H}} = \mathcal{H}_C^N + \tilde{\mathcal{H}}^{(1)} + \tilde{\mathcal{H}}^{(2)} \quad (74)$$

where

$$\begin{aligned} \tilde{\mathcal{H}}^{(1)} &= -\eta^1(\pi - \phi' - \lambda) - \eta^2(\phi'' + 2\lambda' - \pi') \\ \tilde{\mathcal{H}}^{(2)} &= \frac{1}{2}\eta^1\eta^1 + \eta^1\eta^{2'} - \eta^{1'}\eta^2 - \eta^2\eta^{2''}. \end{aligned} \quad (75)$$

First class constraints (73) and Hamiltonian (74) represent a rank zero gauge theory.

6 Gauge Invariant Chiral Schwinger Models

In this section we use our formalism in a theory in which the Δ -matrix has constant elements. The gauge non-invariant bosonized chiral Schwinger model [15, 16], in (1 + 1) dimensions with regularization parameter $a = 1$ is described by the Lagrangian density:

$$\mathcal{L}^N = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (g^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\mu\phi A_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu \quad (76)$$

in which ϕ is a scalar field and A_μ is a vector field. There appear four second class constraints [17]:

$$\begin{aligned} \Theta_1 &\equiv \pi_0 \approx 0 & \Theta_2 &\equiv E' + \phi' + \pi + A_1 \approx 0 \\ \Theta_3 &\equiv E \approx 0 & \Theta_4 &\equiv -\pi - \phi' - 2A_1 + A_0 \approx 0 \end{aligned} \quad (77)$$

where π , π_0 and E are momenta conjugate to ϕ , A_0 , and A_1 respectively. The canonical Hamiltonian density corresponding to Eq. (76) is

$$\mathcal{H}_C^N = \frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2 + \frac{1}{2}E^2 + EA'_0 + (\pi + \phi' + A_1)(A_1 - A_0). \quad (78)$$

It is clear that Eq. (77) represent a second class constrained system with the algebra

$$\{\Theta_i(\mathbf{x}, t), \Theta_j(\mathbf{y}, t)\} = \Delta_{ij}\delta(\mathbf{x} - \mathbf{y}) \quad (79)$$

where

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{pmatrix}. \quad (80)$$

As before we can omit the δ -function and discuss about the discrete part Δ . For construction a first class theory, it is necessary to define four auxiliary fields $\eta^\alpha(x)$ where $\alpha = 1, 2, 3, 4$. In agreement with Eq. (41) we chose them such that

$$\omega^{\alpha\beta} = -\Delta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \end{pmatrix}. \quad (81)$$

Remembering that the trivial choice $\chi = 1$ satisfy (13) and according to (43), the new set of constraint are found to be:

$$\begin{aligned} \tau_1 &\equiv \pi_0 + \eta^1 \approx 0 \\ \tau_2 &\equiv E' + \phi' + \pi + A_1 + \eta^2 \approx 0 \\ \tau_3 &\equiv E + \eta^3 \approx 0 \\ \tau_4 &\equiv -\pi - \phi' - 2A_1 + A_0 + \eta^4 \approx 0 \end{aligned} \quad (82)$$

From Eqs. (47) and (48) the correction terms of the embedded Hamiltonian density are derived. As a result

$$\tilde{\mathcal{H}} = \mathcal{H}_c^{\mathcal{N}} + \tilde{\mathcal{H}}^{(1)} + \tilde{\mathcal{H}}^{(2)} \quad (83)$$

where

$$\tilde{\mathcal{H}}^{(1)} = \eta^1 [2\Theta_3 - \phi'' - \pi' - 2(A_1' + E)] - \eta^2 (2\Theta_2 + \Theta_4) + \eta^3 \Theta_3 - \eta^4 \Theta_2 \quad (84)$$

and

$$\tilde{\mathcal{H}}^{(2)} = 2\eta^1 \eta^1 - \eta^1 \eta^{1''} - \eta^1 \eta^{2'} - \eta^2 \eta^2 + \eta^2 \eta^{1'} - \eta^2 \eta^4 + \frac{1}{2} \eta^3 \eta^3. \quad (85)$$

It can be checked that Eqs. (82) and (83) represent a rank zero gauge invariant theory in the extended phase space.

7 Conclusion

As discussed in the previous sections, BFFT approach is a method for converting a second class constrained system to a first class one which can be quantized according to the usual quantization methods of first class systems; for instance canonical quantization or path integral approach. In this method the series of correction terms for constraints and every

function in phase space, in principle have infinite terms. In the master equation of BFFT method, Eq.(13), there exist arbitrariness for some basic parameters. It is possible to make truncated series for functions of the phase space provided that we chose these parameters in a convenient way. This is done for some special models in Refs. [9, 18], but a systematic method has not been proposed for the general case. However, it seems difficult to truncate series in BFFT method for an arbitrary second class system; i.e. a system in which the elements of Δ -matrix are functions of phase space. We solved this problem when Δ -matrix is in symplectic form or its elements are constants. These cases for Δ -matrix do not lose the generality of the problem. In fact it has been shown that one can convert every second class constrained system to a symplectic system [10, 11]. On the other hand, in most covariant physical models the Δ -matrix has constant elements as we showed for some of them in sections 4-6. The method can be applied to several second class systems in the similar way.

Acknowledgment

The authors thanks F. Loran for useful discussion and comments.

References

- [1] P A M Dirac, " *Lectures on Quantum Mechanics*", Belfer graduate School, Yeshiva Univ. Press, New York, 1964
- [2] L D Faddev, Theo. Math. Phys.**1**, (1970) 1.
- [3] I A Batalin and E S Fradkin, Nucl. Phys. **B 279**, (1987) 514; Phys. Lett. **B 180**, (1986) 157.
- [4] I A Batalin and I V Tyutin, Int. J. Mod. Phys. **A 6**, (1991) 3255.
- [5] E S Fradkin and G A Vilkovisky, Phys. Lett. **B 55**, (1975) 224.
- [6] C Becchi, A Rouet and R Stora, Ann. Phys. (N.Y.) **98**, (1976) 287.
- [7] M Henneaux and C Teitelboim, Phys. Rep. **C 126**, (1985) 1; " *Quantization of Gauge Systems*" Princeton Univ. Press, 1992.
- [8] E C G Stückelberg, Helv. Phys. Act. **30**, (1957) 209.
- [9] N Banerjee, R Banerjee and S Ghosh, Ann. Phys.**241**, (1995) 237.

- [10] F Loran, Phys. Lett. B **554**, (2003) 207.
- [11] F Loran and A Shirzad, Int. J. Mod. Phys. A **17**, (2002) 625.
- [12] H Goldstein, " *Classical Mechanics*" 2nd edition, Addison-Wesley, 1980.
- [13] P P Srivastava, Phys. Rev. Lett. **63**, (1989) 2791.
- [14] J Wess and B Zumino, Phys. Lett. B **37**, (1971) 95.
- [15] R Jackiw and R Rajaraman, Phys. Rev. Lett. **54**, (1985) 1219.
- [16] P Mitra and R Rajaraman, Phys. Lett. B **225**, (1989) 267.
- [17] U Kulshreshtha, D S Kulshreshtha and H J W Müller-Kirsten, in " *Constraint Theory and Quantization Methods*" World Scientific Press, Singapore, 1994, 305
- [18] N Banerjee, R Banerjee and S Ghosh, Phys. Rev. D **49**, (1994) 1996; W Oliveira and J A Neto, Nucl. Phys. B **533**, (1998) 6110; R Banerjee, Phys. Rev. D **48**, (1993) R5467; R Banerjee, H J Rothe and K D Rothe, Phys. Rev. D **49**, (1994) 5438; R Banerjee, H J Rothe, Nucl. Phys. B **447**, (1995) 183; W T Kim and Y J Park, Phys. Lett. B **336**, (1994) 376; R Banerjee and J B Neto, Nucl. Phys. B **499**, (1997) 453; E Harikumar and M Sivakumar, Nucl. Phys. B **565**, (2000) 385.