Gauge Fixing in the Chain by Chain Method

A. Shirzad*   F. Loran†

Department of Physics, Isfahan University of Technology (IUT)
Isfahan, Iran,
Institute for Studies in Theoretical Physics and Mathematics (IPM)
P. O. Box: 19395-5531, Tehran, Iran.

Abstract

In a recent work we showed that for a Hamiltonian system with constraints, the set of constraints can be investigated in first and second class constraint chains. We show here that using this "chain by chain" method for an arbitrary system one can fix the gauges in the most economical and consistent way. We show that it is enough to assume some gauge fixing conditions conjugate to last elements of first class chains. The remaining necessary conditions would emerge from consistency conditions.

1 Introduction

It is well known that gauge theories correspond to Hamiltonian constraint systems with first class constraints. Dirac has conjectured that first class constraints (primary or secondary) are generators of gauge transformations [1]. Despite some counterexamples [2] one can assume the validity of Dirac conjecture under suitable regularity conditions [3]. The presence of first class constraints and the associated gauge freedoms indicates that corresponding to any given physical state there exist some orbit in phase space, i. e. gauge orbit. Gauge transformations translate the system along gauge orbits. One can impose further restrictions on the canonical variables, gauge fixing conditions, to make a one to one correspondence between them and physical states. In this way the initial phase space reduces to a smaller one on which both constraints and gauge fixing conditions (GFC) do vanish. This subspace is called the reduced phase space. There are three properties that a satisfactory set of constraints and GFC's should satisfy:

i) The set of constraints should be regular and irreducible [3].

ii) The GFC's should be accessible. They should intersect the gauge orbits at least once. In addition they should completely fix the gauges.

iii) The GFC's should remain valid during the time i.e. their time derivatives should vanish.

The property (*ii*) is well known. The first and third properties though considered practically[1],

*shirzad@cc.iut.ac.ir

†loran@cc.iut.ac.ir

[1] See for example ref.[4].

but are not emphasized through the literature. In most cases, people work with well-behaved models possessing regular and irreducible constraints and propose suitable GFC's considering the second property mentioned above.

Since first class constraints are generators of gauge transformation the process of gauge fixing strongly depends on the method of producing the constraints. For example, there are some methods which lead to a set of reducible constraints [5, 6]. In these cases one needs primarily a consistent method to distinguish the independent gauge degrees of freedom.

In ref. [7] we proposed a new method, the chain by chain method, for constructing an irreducible set of constraints. In this method constraints are classified in a number of second class and a number of first class constraint chains. This article is devoted to gauge fixing in the chain by chain method. We show that one only needs to find GFC's that fix the gauge freedoms associated to the last element of first class chains. Consistency conditions generate the remaining needed GFC's. In this way the properties $(i) - (iii)$ are satisfied consistently. Moreover, the number of necessary GFC's to be found is just equal to the number of first class chains that in general is less than the number of first class constraints. We do not consider difficulties due to Gribov ambiguities [8] and the problem of covariance of the formalism in this work.

In the following section we review basic concepts of constraint systems and gauge transformations in the extended and total Hamiltonian formalism. The chain by chain method is also reviewed briefly in that section. Our method for gauge fixing in the framework of chain by chain method is proposed in section 3. In section 4 we examine our method in Electrodynamics and Yang-Mills theories. Some concluding remarks are given in section 5.

2 Constraints and Gauges

Consider a dynamical system given by a canonical Hamiltonian $H_c(q,p)$ and a set of primary constraints $\phi_1^a(q,p)$, $a = 1,\ldots,n$. The Hamilton-Dirac equations of motion for an arbitrary function $g(q,p)$ read [1]

$$\dot{g}(q,p) = \{g, H_T\}\,, \tag{1}$$

where

$$H_T = H_c + \sum_a v_a \phi_1^a, \tag{2}$$

in which, v_a are Lagrange multipliers. Equation (1) together with constraint relations $\phi_1^a(q,p) = 0$ can be derived by varying the total action

$$S_T = \int dt\,(\dot{q}_i p_i - H_T)\,, \tag{3}$$

with respect to canonical variables (q_i, p_i) and Lagrange multipliers v_a. Gauge transformations are defined as transformation on phase space trajectories $(q_i(t), p_i(t))$ and Lagrange multipliers that include arbitrary functions of time and leave the total action S_T invariant. In models satisfying Dirac conjecture one can show that gauge transformations transform different classes of solutions, belonging to different choices of arbitrary functions of time, to each other [3].

As is well known consistency conditions for primary constraints, $\dot{\phi}_1^a = 0$, may lead to determination of some Lagrange multipliers or appearing secondary constraints. In the traditional method of producing the secondary constraints, i.e. the level by level method [3, 9, 10, 11, 12], constraints appear in a sequence of levels of irreducible constraints. The primary constraints form the first level. One obtains the constraints of the n-th level,say, by considering the consistency of constraints of the $(n-1)$-th level. By construction no new constraint emerges from the consistency conditions of the last level.

In the chain by chain method, conversely, [7] one investigates the consistency of primary constraints one by one. For primary constraint ϕ_1^a, say, the corresponding chain is knitted via the recursion relation

$$\phi_n^a = \{\phi_{n-1}^a, H_c\}\,. \tag{4}$$

Some chains terminate when a Lagrange multiplier is determined. These are second class chains that contain only second class constraints. The remaining chains, first class chains, which contain only first class constraints, end up when consistency of the last element is achieved identically. The whole algorithm is given in [7]. Following this algorithm one can separate first class and second class constraints from each other and arrange them in the associated chains. In addition constraints in different chains commute with each other, i.e. the Poisson bracket of any element of one chain with any element of other chains vanishes on the surface of the constraints. Therefore the structure of first class chains do not change if one replaces the Poisson brackets with Dirac brackets and eliminates the second class constraints. Consequently one can consider every constraint system as a purely first class system when the question of gauge fixing arises. In the following we study gauge fixing in first class systems. The above observations guarantee the validity of our results in general cases.

3 Gauge Fixing

Consider a system with N first class constraints arranged in m first class chains:

$$\begin{array}{ccccccc}
\phi_1^1 & \phi_1^2 & \dots & \phi_1^a & \dots & \phi_1^m \\
\vdots & \vdots & & \vdots & & \vdots \\
 & & & & & \phi_{N_m}^m \\
\vdots & \phi_{N_2}^2 & & \vdots & & \\
\phi_{N_1}^1 & & & & & \\
 & & & \phi_{N_a}^a & &
\end{array} \tag{5}$$

The evolution of gauge invariant quantities may also be determined by the extended Hamiltonian

$$H_E = H_c + \sum_{a,i} \lambda_i^a \phi_i^a, \tag{6}$$

where λ_i^a are undetermined Lagrange multipliers, which here can be considered as independent gauge parameters. In the extended formalism, corresponding to each first class constraint there exist one Lagrange multiplier to be determined by gauge fixing. Therefore one should impose an equal number of independent gauge fixing conditions as there are first class constraints. The consistency of gauge fixing conditions determines the Lagrange multipliers. The true dynamics of a constrained system, however, is given by the total Hamiltonian defined in Eq.(2). The extended Hamiltonian can be used instead of the total Hamiltonian, provided that one demand after all that Lagrange multipliers corresponding to secondary constraints (and their variations) vanish [3].

For several reasons gauge fixing in the total Hamiltonian formalism requires some care. First, the number of gauges to be fixed is $N = \sum_{a=1}^{m} N_a$, the total number of first class constraints; while the number of Lagrange multipliers to be determined is m, which is usually less than N. Second, the consistency of GFC's may lead to additional constraints that over-determine the system. Third, the (first class) constraints in the total Hamiltonian formalism do not generate independent gauge transformations. It can be shown [5, 6] that there exist $(N-m)$ differential equations among the gauge parameters corresponding to first class constraints. The question arises that "how can one fix the independent gauges in a consistent way?". This can be answered within the framework of the chain by chain method in a simple way as follows.

Considering the set of first class constraints given in Eq.(5), one may find m gauge fixing conditions $\Omega^a_{N_a}$'s with the following property:

$$\{\Omega^a_{N_a}, \phi^b_n\} \approx \eta^a(q,p)\delta^{ab}\delta_{n,N_a} \tag{7}$$

where $\eta^a(q,p)$ are some arbitrary functions which should not vanish on the surface of the constraints. In principle the set of first class constraints $\phi^a_{N_a}$'s can be considered as a set of momenta. In such an idealized system the gauge fixing conditions $\Omega^a_{N_a}$'s are the corresponding conjugate coordinates and consequently η^a's become proportional to the unity. Therefore, the existence of η^a's can always be assumed.

We show that the remaining GFC's needed to fix the gauge completely can be obtained by using the consistency of $\Omega^a_{N_a}$'s. Since $\{\Omega^a_{N_a}, \phi^b_1\} \approx 0$, the consistency of Ω_{N_a}'s i.e. $\dot{\Omega}_{N_a} = 0$, gives a new set of GFC's as:

$$\Omega^a_{N_a-1} \equiv \{\Omega^a_{N_a}, H_c\}. \tag{8}$$

Let us consider the Poisson bracket of $\Omega^a_{N_a-1}$ with the constraints:

$$\begin{aligned} \{\Omega^a_{N_a-1}, \phi^b_n\} &= \{\{\Omega^a_{N_a}, H_c\}, \phi^b_c\} \\ &= \{H_c, \{\phi^b_n, \Omega^a_{N_a}\}\} - \{\Omega^a_{N_a}, \phi^b_{n+1}\} \end{aligned} \tag{9}$$

where we have used Eq.(4) in the last line. Using Eq.(7) the above expression vanishes for $a \neq b$, as well as for $a = b$ and $n < N_a - 1$. Note specially that the Poisson brackets of $\Omega^a_{N_a-1}$ with the primary constraints vanishes. For $a = b$ and $n = N_a - 1$ Eq.(9) gives:

$$\{\Omega^a_{N_a-1}, \phi^a_{N_a-1}\} \approx -\eta^a(q,p). \tag{10}$$

Consistency of $\Omega^a_{N_a-1}$ leads to $\Omega^a_{N_a-2} \equiv \{\Omega^a_{N_a-1}, H_c\}$ and so on. The generic terms for the GFC's are related to each other as follows:

$$\Omega^a_n = \{\Omega^a_{n+1}, H_c\}, \qquad n = 1, \ldots, N_a - 1 \tag{11}$$

Comparing Eq.(11) with Eq.(4) one realizes that the chains of GFC's are exactly the "mirror images" of the constraint chains, i.e. they are knitted in the opposite direction. The whole story goes on as follows: one begins with ϕ^a_1, goes through consistency conditions until reaches $\phi^a_{N_a}$, then fixes the gauge by finding $\Omega^a_{N_a}$ conjugate to $\phi^a_{N_a}$, turns all the way round through consistency conditions to reach Ω^a_1 at the end point. The story sounds more interesting by repeating the calculations given in Eq.(9) to get:

$$\begin{array}{ll} \{\Omega^a_n, \phi^b_{n'}\} \approx 0 & a \neq b \\ \{\Omega^a_n, \phi^a_{n'}\} \approx 0 & n' < n \\ \{\Omega^a_n, \phi^a_n\} \approx (-1)^{N_a-n}\eta^a(q,p) & \end{array} \tag{12}$$

As is observed each Ω^a_n is really conjugate to its partner ϕ^a_n. The story ends when one investigates the consistency of Ω^a_1's where the Lagrange multipliers are determined due to non-vanishing Poisson brackets

$$\{\Omega^a_1, \phi^a_1\} = (-1)^{N_a-1}\eta^a(q,p).$$

Using Eqs.(12) the matrix of Poisson brackets of constraints with GFC's can be obtained as follows:

$$\begin{pmatrix} \begin{pmatrix} \eta^1 & & 0 \\ \ddots & \ddots & \\ \ddots & \ddots & e_1\eta^1 \end{pmatrix} & & & \mathbf{0} \\ & \begin{pmatrix} \eta^2 & & 0 \\ \ddots & \ddots & \\ \ddots & \ddots & e_2\eta^2 \end{pmatrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{pmatrix} \eta^m & & 0 \\ \ddots & \ddots & \\ \ddots & \ddots & e_m\eta^m \end{pmatrix} \end{pmatrix} \tag{13}$$

where $e_a = (-1)^{N_a-1}$. As is obvious, the determinant of the matrix given in (13) is proportional to $\prod_a [\eta^a(q,p)]^{N_a} \neq 0$. Since chain by chain method guarantees that the set of first class constraints ϕ^a_n's is irreducible this result ensures that the above Ω^a_n, completely fix the gauges [3]. Each non-vanishing block in the matrix (13) corresponds to a definite constraint chain. There emerge indeed some non-vanishing elements below the diameter coming from $\{\Omega^a_n, \phi^a_{n'}\}$ with $n' > n$. One can redefine constraints and GFCs properly to make these off diagonal elements vanish (see [7])

4 Electrodynamics with source and Yang-Mills

As a first example of applying the method let us consider electrodynamics with bosonic source given by the Lagrangian:

$$L = \int d^3x\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|(\partial_\mu + igA_\mu)\Phi|^2 - v(\Phi\Phi^*)\} \tag{14}$$

where $V(\Phi\Phi^*)$ is a potential and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{15}$$

Rewriting L in terms of the dynamical fields $A^\mu(x,t)$, $\eta(x,t)$ and $\psi(x,t)$ where

$$\Phi(x,t) = \eta(x,t)e^{i\psi(x,t)}, \tag{16}$$

the canonical momenta are

$$\Pi^\mu = -F^{0\mu}, \qquad \pi_\eta = \dot{\eta}, \qquad \pi_\psi = \eta^2(\dot{\psi} + gA_0). \tag{17}$$

It is obvious from Eq.(15) that $\phi_1 = \Pi_0$ is our primary constraint. Then the total Hamiltonian can be written as

$$\begin{aligned} H_T = \textstyle\int d^3x\{ & \mathcal{H}^{ED} + \tfrac{1}{2}\pi_\eta^2 + \tfrac{1}{2\eta^2}\pi_\psi^2 - gA_0\pi_\psi \\ & + \tfrac{1}{2}\eta^2(\partial_k\psi)(\partial_k\psi) + \tfrac{1}{2}(\partial_k\eta)(\partial_k\eta) + g\eta^2 A_k(\partial_k\psi + \tfrac{1}{2}gA_k) \\ & + V(\eta) + v(x,t)\Pi^0(x,t)\} \end{aligned} \tag{18}$$

where $v(x,t)$ is the Lagrange multiplier (field) and

$$\mathcal{H}^{ED} = \frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}F_{ij}F_{ij} - A_0\partial_i\Pi_i. \tag{19}$$

We have ignored a surface term in Eq.(19) due to boundary conditions. The secondary constraint serves as

$$\phi_2 = \{\Pi^0, H_T\} = \partial_i\Pi_i + g\pi_\psi. \tag{20}$$

No further constraints emerges since $\{\phi_2, H_T\} = 0$. There is just one constraint chain with two elements.

To fix the gauge one should begin with Ω_2 conjugate to ϕ_2. A simple choice is the Coulomb gauge $\Omega_2 = \partial_i A_i$. Consistency condition of Ω_2 then gives another GFC as

$$\Omega_1 = \{\Omega_2, H_c\} = \partial_i\Pi_i + \partial_i\partial_i A_0. \tag{21}$$

Using Eq.(20) one has $\partial_i\Pi_i \approx g\pi_\psi$, hence from Eq.(21) the scalar potential A^0 is determined in this gauge to be

$$A^0(x,t) = \int d^3y \frac{g\pi_\psi(y,t)}{|x-y|}. \tag{22}$$

One important point to be noted is that if one has imposed the famous gauges $\Omega_2 = \partial_i A_i$ and $\Omega_1 = A_0$ then the consistency condition $\dot{\Omega}_2 = 0$ would over-determine the system by imposing $\pi_\psi = 0$.

As a second example consider pure Yang-Mills theory given by:

$$L = -\frac{1}{4}\int d^3x \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{23}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[\Lambda^\mu, \Lambda^\nu] \tag{24}$$

The dynamical fields $A_\mu^a(x,t)$ are implemented as

$$A^\mu = \Lambda_a^\mu \Lambda^a \tag{25}$$

where Λ_a's are generators of a Lie algebra with structure constants C_{ab}^c:

$$[\Lambda^a, \Lambda^b] = iC_c^{ab}\Lambda^c. \tag{26}$$

The canonical momenta are $\Pi_\mu^a = -F_{0\mu}^a$, where $\phi_1^a = -\Pi_0^a$ serves as the set of primary constraints. The canonical Hamiltonian is

$$H_c = \int d^3x\{\frac{1}{2}\Pi_i^a\Pi_i^a - A_0^a\partial_i\Pi_i^a + gA_0^aA_0^bC_c^{ab}\Pi_i^c + \frac{1}{4}F_{ij}^aF_{ij}^a\} \tag{27}$$

where a surface term is ignored. The total Hamiltonian is

$$H_T = H_c + \int d^3x v^a(x,t)\Pi_0^a(x,t). \tag{28}$$

The secondary constraints follow from the consistency of primary constraints as:

$$\phi_2^a(x,t) = \{\Pi_0^a, H_T\} \approx \partial_i\Pi_i^a - gA_i^bC_{bc}^a\Pi_i^c. \tag{29}$$

As in electrodynamics, one may choose the first set of GFC's as

$$\Omega_2^a = \partial_i A_i^a \approx 0. \tag{30}$$

Consistency of this gauge leads to

$$\Omega_1^a \equiv \{\Omega_2^a, H_c\} \approx \partial_i\Pi_i^a + M_b^aA_0^b \approx 0 \tag{31}$$

where

$$M_b^a = \delta_b^a\partial_i\partial_i + gC_{bc}^aA_i^c\partial_i. \tag{32}$$

To see what is the consequence of imposing the GFC's $\Omega_1^a \approx 0$ on A_0^a's, let define the Green function G_c^b due to operator M_b^a:

$$M_b^a(x)G_c^b(x,y) = \delta_c^a\delta(x-y). \tag{33}$$

Eq.(31) can be solved:

$$A_0^a(x,t) = -\int d^3y\partial_i\Pi_i^b(y,t)G_b^a(x,y) = H^a(x,t). \tag{34}$$

We observe again that the famous gauges $A_0^a \approx 0$ and $\partial_iA_i^a \approx 0$, over-determine the system by imposing an additional condition $H^a(x,t) \approx 0$.

5 Conclusion

Chain by chain method provides a simple constraint structure. In this method the constraints are irreducible. Each Constraint belongs to a chain that is identified by one of the primary constraints. Some chains possess only second class and others possess only first class constraints. Constraints in different chains have vanishing Poisson brackets and constraints belonging to each chain satisfy the recursion relation given in Eq.(4). This structure provides a simple and consistent method for gauge fixing. One searches for a set of constraints that eliminate the gauge freedom associated to the last elements of first class chains. One obtains the remaining necessary gauge fixing conditions by imposing consistency conditions. In this method gauge freedom associated to first class constraints belonging to each first class chain is fixed indeed by only one gauge fixing condition. This is exactly the case in the Lagrangian formalism. Given a Lagrangian, one may fix the gauge partly by adding some proper terms to the Lagrangian. Switching to the Hamiltonian formalism the corresponding primary first class constraints disappear and consequently the related first class chains would not emerge. In other words every gauge fixing term that is added to the Lagrangian fixes the gauge freedom associated to one first class chain. This confirms our method for gauge fixing.

References

[1] P.A. M. Dirac, Can. J. Math. **2**, (1950) 129 ; Proc. R. Soc. London Ser. A **246**, (1958) 326; *"Lectures on Quantum Mechanics"* New York: Yeshiva University Press, 1964,

[2] R. Cawly, Phys. Rev. Lett. **42**, (1979) 413; R. Cawly, Phys. Rev. **D21**, (1980) 2988.

[3] M. Henneaux and C. Teitelboim *"Quantization of Gauge System"* Princeton University Press, Princeton, New Jersey, 1992.

[4] V. M. Villanueva, J. Govaerts, J-L. Lucio-Martinez, J.Phys. A**33**, (2000) 4183.

[5] A. Shirzad A and M. Shabani Moghaddam, *Jour. Phys. A: Math. Gen.* **32**, (1999) 8185.

[6] A. Cabo, M. Chaichian and D. Louis Martinez *J. Math. Phys.* **34**, (1993) 5646.

[7] F. Loran and A. Shirzad, Int. J. Mod. Phys. **A17** (2002) 625.

[8] V. N. Gribov, Nucl. Phys. **B139**, 1.

[9] C. Batlle, J. Gomis , J. M. Pons and N. Roman Roy, *Jour. Math. Phys.* bf 27, (1986) 2953.

[10] M. Henneaux, C. Teitelboim and J. Zanelli, *Nucl. Phys.* **B332**, (1990) 169.

[11] J. Gomis, M. Henneaux and J. M. Pons, *Class. Quantum Grav.* **7**, (1990) 1089.

[12] X. Gracia and J. M. Pons, *Jour. Phys. A: Math. Gen.* **25**, (1992) 6357.