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The BFT method with chain structure

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Abstract

We have constructed a modified BFT method that preserves the chain structure of constraints. This method has two advantages: first, it leads to less number of primary constraints such that the remaining constraints emerge automatically; and second, it gives less number of independent gauge parameters. We have applied the method for bosonized chiral Schwinger model. We have constructed a gauge invariant embedded Lagrangian for this model.

Dirac as a pioneer, quantized classical gauge theories by converting Poisson brackets to quantum commutators [1]. However, for second class constraint systems it is necessary to replace Poisson brackets by Dirac brackets and then convert them to quantum commutators. Sometimes this process implies problems such as factor ordering which makes this approach improper. The BFT method, however, solves this ambiguity by embedding the phase space in a larger space including some auxiliary fields [2,3]. In this way one can convert second class constraints to first class ones and then apply the well-known quantization method of gauge theories [4,5]. In our previous paper [6] we showed that if one chooses arbitrary parameters of the BFT method suitably then the power series of auxiliary fields for the embedded constraints and Hamiltonian could be truncated in some cases.

In this Letter we want to preserve the *chain structure* of a second class system (except for the last element of the chain) during the BFT embedding. The main idea of the chain structure, as fully discussed in [7], is that it is possible to derive the constraints as commuting distinct chains such that within each chain the following iterative relation holds

$$\Phi_\alpha^a = \left\{\Phi_{\alpha-1}^a, H_c\right\}, \tag{1}$$

where Φ_0^a stand for primary constraints. The advantages of this method will be discussed afterward.

Consider a second class constraint system described by the Hamiltonian H_0 and a set of second class constraints Φ_α; $\alpha = 1, \dots, N$ satisfying the algebra

$$\Delta_{\alpha\beta} = \{\Phi_\alpha, \Phi_\beta\}, \tag{2}$$

where Δ is an antisymmetric and invertible matrix. For simplicity and without loss of generality we suppose that the second class constraints Φ_α are elements of one chain. The results can be extended to multi-chain

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system by adding the chain index (like the superscript a in Eq. (1)) to the constraints Φ_α. For converting this second class system into a gauge system, one can enlarge the phase space by introducing auxiliary variables η where we *assume* their algebra to be

$$\omega^{\alpha\beta} = \{\eta^\alpha, \eta^\beta\}. \tag{3}$$

We demand that the embedded constraints $\tau_\alpha(q, p, \eta)$ and Hamiltonian $\tilde{H}(q, p, \eta)$ in the extended phase space satisfy the following algebra

$$\{\tau_\alpha, \tau_\beta\} = 0, \tag{4}$$

$$\{\tau_\alpha, \tilde{H}\} = \tau_{\alpha+1}, \quad \alpha = 1, \ldots, N-1, \tag{5}$$

$$\{\tau_N, \tilde{H}\} = 0. \tag{6}$$

This gives an Abelian first class chain such that its terminating element commute with the Hamiltonian. We call this system a *semi-strongly involutive system*; compared with strongly involutive one in which the constraints commute with the Hamiltonian.

As discussed in [2–4], considering the power series

$$\tau_\alpha = \sum_{n=0}^{\infty} \tau_\alpha^{(n)}, \quad \tau_\alpha^{(n)} \sim \eta^n, \tag{7}$$

$$\tilde{H} = \sum_{n=0}^{\infty} H^{(n)}, \quad H^{(n)} \sim \eta^n, \tag{8}$$

in which $\tau_\alpha^{(0)} = \Phi_\alpha$ and $H^{(0)} = H_c(q, p)$, one can show that these may be solutions to Eqs. (4) and (5) if

$$\tau_\alpha^{(1)} = \chi_{\alpha\beta}(q, p)\eta^\beta, \tag{9}$$

$$\tau_\alpha^{(n+1)} = -\frac{1}{n+2}\eta^\beta \omega_{\beta\gamma}\chi^{\gamma\delta}B_{\delta\alpha}^{(n)}, \quad n \geqslant 1, \tag{10}$$

$$H^{(n+1)} = -\frac{1}{n+1}\eta^\alpha \omega_{\alpha\beta}\chi^{\beta\gamma}\Lambda_\gamma^{(n)} \tag{11}$$

in which

$$B_{\alpha\beta}^{(1)} = \{\tau_{[\alpha}^{(0)}, \tau_{\beta]}^{(1)}\}_{(\eta)}, \tag{12}$$

$$\begin{aligned} B_{\alpha\beta}^{(n)} &= \frac{1}{2}B_{[\alpha\beta]} \\ &= \sum_{m=0}^{n}\{\tau_\alpha^{(n-m)}, \tau_\beta^{(m)}\} \\ &\quad + \sum_{m=0}^{n-2}\{\tau_\alpha^{(n-m)}, \tau_\beta^{(m+2)}\}_{(\eta)}, \quad n \geqslant 2, \end{aligned} \tag{13}$$

$$\Lambda_\alpha^{(0)} = \{\tau_\alpha^{(0)}, H^{(0)}\} - \tau_{\alpha+1}^{(0)}, \quad \alpha < N, \tag{14}$$

$$\begin{aligned} \Lambda_\alpha^{(1)} &= \{\tau_\alpha^{(1)}, H^{(0)}\} + \{\tau_\alpha^{(0)}, H^{(1)}\} + \{\tau_\alpha^{(2)}, H^{(1)}\}_{(\eta)} \\ &\quad - \tau_{\alpha+1}^{(1)}, \quad \alpha < N, \end{aligned} \tag{15}$$

$$\begin{aligned} \Lambda_\alpha^{(n)} &= \sum_{m=0}^{n}\{\tau_\alpha^{(n-m)}, H^{(m)}\} \\ &\quad + \sum_{m=0}^{n-2}\{\tau_\alpha^{(n-m)}, H^{(m+2)}\}_{(\eta)} \\ &\quad + \{\tau_\alpha^{(n+1)}, H^{(1)}\}_{(\eta)} - \tau_{\alpha+1}^{(n)}, \quad n \geqslant 2,\ \alpha < N \end{aligned} \tag{16}$$

and $\chi_{\alpha\beta}(q, p)$ should satisfy the following equation

$$\Delta_{\alpha\beta} + \chi_{\alpha\gamma}\omega^{\gamma\lambda}\chi_{\beta\lambda} = 0. \tag{17}$$

For $\alpha = N$, due to Eq. (6), the last terms in Eqs. (14)–(16) are absent. The above results for semi-strongly involutive system is different from our previous results for strongly involutive one [2–4] in terms $\tau_{\alpha+1}^{(0)}$, $\tau_{\alpha+1}^{(1)}$ and $\tau_{\alpha+1}^{(n+1)}$ in Eqs. (14)–(16), respectively. We remind that [6], the aim is to choose a suitable solution for $\chi_{\alpha\beta}$ in Eq. (17) such that the series for τ_α and $\tilde{H}$ truncates after a few steps. For example when Δ is the symplectic matrix J, the choice $\omega = -J$ and $\chi = J$ solves (17); similarly when Δ is a constant (antisymmetric) matrix, the choice $\omega = -\Delta$ and $\chi = 1$ is appropriate.

Now let see what is the advantage of the chain structure in our modified BFT method. We emphasize on two points.

(1) Suppose we are given a singular Lagrangian L which leads to one primary constraint Φ_1. Let the secondary constraints emerge as a second class chain with elements Φ_α $(\alpha = 1, \ldots, N)$ resulting from consistency of the constraints implicit in the chain relation (1). After embedding one finds in the traditional BFT method a Hamiltonian $\tilde{H}$ together with N constraints, all in strongly involution. The constraints can be viewed in this case as N *given primary constraints*. However, preserving the chain structure, one ultimately obtains an embedded Hamiltonian with just one primary constraint. The other $N-1$ constraints are then obtained automatically from the consistency conditions.

It should be added that sometimes it is possible to reconstruct a singular Lagrangian from a given canonical Hamiltonian and primary constraints, even though it is not guarantied generally. However, the less the number of primary constraints, the more the chance to find the original Lagrangian which gives the desired primary constraints and Hamiltonian. In the following we will give an example to show this point. We think that our modified BFT method improves the chance of finding a corresponding Lagrangian yielding the embedded primary constraints.

To be more precise, when the Hamiltonian is quadratic and the primary constraints are linear with respect to the phase space coordinates, one can easily reconstruct the corresponding singular Lagrangian. To do this, one should solve the constraint equations for a number of momenta; and then insert just linear terms with respect to the corresponding velocities (with coefficients given by the solutions of the momenta) in the Lagrangian. The remaining quadratic terms of the Lagrangian can be found from the corresponding terms of the Hamiltonian in a regular way.

The important point is that for the cases considered in this Letter ($\varDelta = J$ or $\varDelta =$ constant) the constraints are necessarily linear before embedding. As explained in more details in [6] after embedding, the constraints remain linear with respect to the coordinates of the extended phase space. Moreover, if the original Hamiltonian is quadratic, it would remain quadratic after embedding. It is clear that beginning with a quadratic singular Lagrangian (which is the case for most interesting models) a number of linear primary (as well as secondary) constraints and a quadratic Hamiltonian emerge. Therefore, using our method, after embedding one finds a quadratic embedded Hamiltonian together with some linear primary constraints in the same number as the original model. This guarantees that one can find the embedded singular (Wess–Zumino) Lagrangian, with the primary constraints of the embedded model as the primary constraints.

(2) When a chain structure exists the number of independent gauge parameters is much less than when we lack it. By "gauge parameters" we mean arbitrary functions of time that appear in the solutions of equations of motion. To be more precise, in the traditional BFT method where we have ultimately the strongly involutive constraints τ_α and Hamiltonian $\tilde{H}$, it is clear that the following function acts as the generating function of gauge transformation

$$G = \sum \varepsilon^\alpha(t)\tau_\alpha(q, p, \eta). \tag{18}$$

As is apparent, here the number of gauge parameters $\varepsilon^\alpha(t)$ is equal to the total number of constraints.

However, in the presence of the chain structure it can be shown that the number of gauge parameters is just equal to the number of distinct chains [8]. There exist two equivalent methods to construct gauge generating function [9,10]. In the special case, where all the constraints are Abelian and the terminating elements of each chain commute strongly with Hamiltonian, the gauge generating function can be written as

$$G = \sum_{i=1}^{m}\sum_{\alpha=1}^{N_i}\left(-\frac{d}{dt}\right)^{N_i-\alpha}\zeta_i(t)\varPhi_\alpha^{(i)}, \tag{19}$$

where $\zeta_i(t)$ are infinitesimal arbitrary functions of time, m is the total number of first class chains and N_i is the length of the ith chain.

It should be remembered that the generating function (18) gives the symmetries of the extended action, while G in (19) gives the symmetries of the total action. The former formalism (i.e., the extended Hamiltonian), however, may be shown to give the correct results for physical (non-gauged) observables.

Now it is instructive to apply the general idea discussed above to a specific model. The bosonized chiral Schwinger model in $1+1$ dimensions with regularization parameter $a = 1$ is described by the Lagrangian density [11,12]:

$$\mathcal{L}^N = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \left(g^{\mu\nu} - \varepsilon^{\mu\nu}\right)\partial_\mu\phi A_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu \tag{20}$$

in which ϕ is a scalar and A_μ is a vector field. There is one second class chain including four second class constraints as follows:

$$\begin{aligned}
&\varPhi_1 \equiv \pi_0 \approx 0,\\
&\varPhi_2 \equiv E' + \phi' + \pi + A_1 \approx 0,\\
&\varPhi_3 \equiv E \approx 0,\\
&\varPhi_4 \equiv -\pi - \phi' - 2A_1 + A_0 \approx 0,
\end{aligned} \tag{21}$$

where π, π_0 and E are momenta conjugate to ϕ, A_0, and A_1, respectively. The canonical Hamiltonian density corresponding to Eq. (20) is

$$\mathcal{H}_C^N = \frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2 + \frac{1}{2}E^2 + EA_0' + (\pi + \phi' + A_1)(A_1 - A_0). \tag{22}$$

Eqs. (21) represent a second class constrained system with the algebra

$$\{\Phi_i(\mathbf{x}, t), \Phi_j(\mathbf{y}, t)\} = \Delta_{ij}\delta(\mathbf{x} - \mathbf{y}), \tag{23}$$

where

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{pmatrix}. \tag{24}$$

Let define four auxiliary fields $\eta^\alpha(x)$ with the algebra given by $\omega = -\Delta$. As discussed above, the choice $\chi = 1$ satisfy Eq. (17). Then following the instructions given in Eqs. (9) and (10), the new set of first class constraint are found to be

$$\begin{aligned} &\tau_1 \equiv \pi_0 + \eta^1 \approx 0, \\ &\tau_2 \equiv E' + \phi' + \pi + A_1 + \eta^2 \approx 0, \\ &\tau_3 \equiv E + \eta^3 \approx 0, \\ &\tau_4 \equiv -\pi - \phi' - 2A_1 + A_0 + \eta^4 \approx 0. \end{aligned} \tag{25}$$

From Eqs. (14)–(16) the embedded Hamiltonian which preserves the chain structure (except for the last element) is

$$\tilde{\mathcal{H}} = \mathcal{H}_c^N + \mathcal{H}^{(1)} + \mathcal{H}^{(2)}, \tag{26}$$

where

$$\mathcal{H}^{(1)} = -\eta^1(\phi'' + \pi' + 2A_1' + 2E), \tag{27}$$

$$\mathcal{H}^{(2)} = \eta^{1\prime}\eta^2 + \eta^2\eta^4 + \eta^2\eta^2 - \eta^1\eta^{1\prime\prime} - 2\eta^1\eta^3 - \frac{1}{2}\eta^3\eta^3. \tag{28}$$

Here it can be checked that the constraints in Eqs. (25) satisfy the chain structure relation (1) where τ_1 is the primary constraint. The chiral Schwinger model has also been discussed in [6] where the set of first class constraints are the same as (25) while the embedded Hamiltonian $\tilde{H}$ is different. It is worth nothing that this Hamiltonian does not differ from that of the strongly involutive formulation by the mere addition of a suitable combination of the first class constraints.

According to Eq. (19) the gauge generating function written in terms of just one infinitesimal gauge parameter $\zeta(x, t)$ is

$$G = \int (-\zeta\tau_4 + \dot{\zeta}\tau_3 - \ddot{\zeta}\tau_2 + \dddot{\zeta}\tau_1)\,dx. \tag{29}$$

The infinitesimal gauge variations of the original and auxiliary fields generated by G are as follows

$$\begin{aligned} &\delta\phi = \zeta - \ddot{\zeta}, && \delta A_0 = \dddot{\zeta}, \\ &\delta A_1 = \partial\ddot{\zeta} + \dot{\zeta}, && \delta\pi = \partial\zeta - \partial\ddot{\zeta}, \\ &\delta\pi_0 = \zeta, && \delta E = \ddot{\zeta} - 2\zeta, \\ &\delta\eta^1 = -\zeta, && \delta\eta^2 = -\dot{\zeta}, \\ &\delta\eta^3 = 2\zeta - \ddot{\zeta}, && \delta\eta^4 = 2\dot{\zeta} - \dddot{\zeta}. \end{aligned} \tag{30}$$

It can be directly checked that the total action is invariant under these variations.

We can redefine the auxiliary fields η^1 to η^4 into two fields η, ξ and their canonical momenta π_η and π_ξ as follows

$$\begin{aligned} &\eta = \eta^1, && \pi_\eta = \eta^4 + 2\eta^2, \\ &\xi = \eta^3, && \pi_\xi = \eta^2. \end{aligned} \tag{31}$$

Fortunately all the terms in Hamiltonian (26) are quadratic. This make it easy to reconstruct the following Lagrangian

$$\mathcal{L} = \mathcal{L}^N + \eta(\phi'' + \pi^1 + 2A^1 + 2E) + \left(\dot{\eta}^2 - \eta'^2 - \dot{\eta}\eta' + \dot{\eta}\dot{\xi} + 2\eta\xi + \frac{1}{2}\xi^2\right) - \eta\dot{A}^0. \tag{32}$$

The first term is the original Lagrangian (20), the second and third terms are due to $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ in Eqs. (27) and (28) respectively and the last term $(-\eta\dot{A}^0)$ is the crucial term which converts the primary constraint π_0 into $\pi_0 + \eta$ of the embedded system. One can directly check that beginning with the Lagrangian (32) the first class system given by Hamiltonian $\tilde{H}$ in Eq. (26) and τ_1 to τ_4 in Eq. (25) would be obtained.

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